

FUNCTIONAL EQUATION FOR THE SELMER GROUP OF NEARLY ORDINARY HIDA DEFORMATION OF HILBERT MODULAR FORMS.

SOMNATH JHA, DIPRAMIT MAJUMDAR

ABSTRACT. We establish a duality result proving the ‘functional equation’ of the characteristic ideal of the Selmer group associated to a nearly ordinary Hilbert modular form over the cyclotomic \mathbb{Z}_p extension of a totally real number field. Further, we use this result to establish a duality or algebraic ‘functional equation’ for the ‘big’ Selmer groups associated to the corresponding nearly ordinary Hida deformation. The multivariable cyclotomic Iwasawa main conjecture for nearly ordinary Hida family of Hilbert modular forms is not established yet and this can be thought of as an evidence to the validity of this Iwasawa main conjecture. We also prove a functional equation for the ‘big’ Selmer group associated to an ordinary Hida family of elliptic modular forms over the \mathbb{Z}_p^2 extension of an imaginary quadratic field.

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INTRODUCTION

We fix an odd rational prime p and N a natural number prime to p . Throughout, we fix an embedding ι_∞ of a fixed algebraic closure $\bar{\mathbb{Q}}$ of \mathbb{Q} into \mathbb{C} and also an embedding ι_ℓ of $\bar{\mathbb{Q}}$ into a fixed algebraic closure $\bar{\mathbb{Q}}_\ell$ of the field \mathbb{Q}_ℓ of the ℓ -adic numbers, for every prime ℓ . Let F denote a totally real number field and K denote an imaginary quadratic field. For any number field L , S_L will denote a finite set of places of L containing the primes dividing Np . The cyclotomic \mathbb{Z}_p extension of L will be denoted by L_{cyc} and the unique \mathbb{Z}_p^2 extension of K will be denoted by K_∞ . Set $\Gamma := \text{Gal}(L_{\text{cyc}}/L) \cong \mathbb{Z}_p$ and $\Gamma_K := \text{Gal}(K_\infty/K) \cong \mathbb{Z}_p^2$. Let B be a commutative, complete, noetherian, normal, local ring of characteristic 0 with finite residue field of characteristic p . We will denote by $B[[\Gamma]]$ (resp. $B[[\Gamma_K]]$) the Iwasawa

algebra of Γ (resp. Γ_K) with coefficient in B . Let M be a finitely generated torsion $B[[\Gamma]]$ (resp. $B[[\Gamma_K]]$) module. Then M^ι denote the $B[[\Gamma]]$ (resp. $B[[\Gamma_K]]$) module whose underlying abelian group is the same as M but the Γ (resp. Γ_K) action is changed via the involution sending $\gamma \mapsto \gamma^{-1}$ for every $\gamma \in \Gamma$ (resp. $\gamma \in \Gamma_K$). We denote the characteristic ideal of M in $B[[\Gamma]]$ (resp. $B[[\Gamma_K]]$) by $Ch_{B[[\Gamma]]}(M)$ (resp. $Ch_{B[[\Gamma_K]]}(M)$). The main results of the article is the following theorem.

Theorem (Theorem 3.10 and Theorem 4.9). *Let \mathcal{R} be a branch of Hida's universal nearly ordinary Hecke algebra associated to nearly ordinary Hilbert modular forms (resp. Hida's universal ordinary Hecke algebra associated to ordinary elliptic modular forms). Let $\mathcal{T}_{\mathcal{R}}$ be a G_F (resp. $G_{\mathbb{Q}}$) invariant lattice associated to 'big' Galois representation $(\rho_{\mathcal{R}}, \mathcal{V}_{\mathcal{R}})$ and set $\mathcal{T}_{\mathcal{R}}^* := \text{Hom}_{\mathcal{R}}(\mathcal{T}_{\mathcal{R}}, \mathcal{R}(1))$. Let $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})$ (resp. $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/K_{\infty})$) denote the dual Selmer group of \mathcal{R} over F_{cyc} (resp. K_{∞}). Then under certain conditions, as ideals in $\mathcal{R}[[\Gamma]]$ (resp. $\mathcal{R}[[\Gamma_K]]$), we have the equality*

$$Ch_{\mathcal{R}[[\Gamma]]}(\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})) = Ch_{\mathcal{R}[[\Gamma]]}(\mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/F_{\text{cyc}})^\iota)$$

(resp.

$$Ch_{\mathcal{R}[[\Gamma_K]]}(\mathcal{X}(\mathcal{T}_{\mathcal{R}}/K_{\infty})) = Ch_{\mathcal{R}[[\Gamma_K]]}(\mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/K_{\infty})^\iota). \quad \square$$

Theorem 3.10 and Theorem 4.9 generalizes [J-P, Theorem 5.2]. An ingredient in the proof of Theorem 3.10 is Theorem 2.10, where we prove a functional equation for the Selmer group of a single Hilbert modular form. Theorem 2.10 is a variant of [Gr1, Theorem 2] and [Pe, Theorem 4.2.1]. The motivation for the above theorems is explained below.

Let E be an elliptic curve defined over \mathbb{Q} with good ordinary reduction at an odd prime p . Let L_E be the complex L function of E . Then by modularity theorem, E is modular and L_E coincides with the L -function of a weight 2 newform of level N_E , where N_E is the conductor of E . Moreover, L_E has analytic continuation to all of \mathbb{C} and if we set

$$\Lambda_E(s) := N_E^{s/2} (2\pi)^{-s} \Gamma(s) L_E(s)$$

to be the completed L -function of E , where $\Gamma(s)$ is the usual Γ function, then (due to Hecke)

$$\Lambda_E(2-s) = \pm \Lambda_E(s), s \in \mathbb{C}. \quad (1)$$

Thus L_E satisfies a functional equation connecting values at s and $2-s$, where $s \in \mathbb{C}$. Let ϕ vary over the Dirichlet characters of $\Gamma = \text{Gal}(\mathbb{Q}_{\text{cyc}}/\mathbb{Q})$ i.e. $\phi \in \hat{\Gamma}$, then the twisted L -function $L_E(s, \phi)$, will also satisfy a functional equation similar to (1) connecting the values of $L_E(2-s, \phi)$ and $L_E(s, \phi^{-1})$. In particular, $L_E(1, \phi)$ and $L_E(1, \phi^{-1})$ are related by a functional equation.

Now, by the work of Mazur and Swinnerton-Dyer [M-SD], the p -adic L -function of E exists. Let $g_E(T)$ be the power series representation of the p -adic L -function of E in $\mathbb{Z}_p[[\Gamma]] \otimes \mathbb{Q}_p \cong \mathbb{Z}_p[[T]] \otimes \mathbb{Q}_p$. Then we have,

- $g_E(0) = (1 - \alpha_p^{-1})^2 L_E(1) / \Omega_E$,
- $g_E(\phi(T) - 1) = \frac{L_E(1, \phi) \beta_p^n}{\tau(\phi) \Omega_E}$, for a character ϕ of Γ of order $p^n \geq 1$. Here $\alpha_p + \beta_p = a_p$, $\alpha_p \beta_p = p$ with $p \nmid \alpha_p$, Ω_E is the real period of E and $\tau(\phi)$ is the Gauss sum of ϕ .

We assume that $g_E \in \mathbb{Z}_p[[\Gamma]]$. Then using the above interpolation properties, we can deduce from (1), a functional equation for the p -adic L -function (also proven in [M-T-T]), given by

$$g_E(T) = u_E g_E\left(\frac{1}{1+T} - 1\right), \quad (2)$$

where u_E is a unit in the ring $\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$. In other words, we have an equality of ideals in $\mathbb{Z}_p[[\Gamma]]$,

$$(g_E(T)) = (g_E\left(\frac{1}{1+T} - 1\right)). \quad (3)$$

By the cyclotomic Iwasawa main conjecture for E (cf. [S-U, § 3.5]), the p -adic L -function $g_E(T)$ is a characteristic ideal of the (dual) p^∞ -Selmer group $X(T_p E / \mathbb{Q}_{\text{cyc}})$. Thus, we should get

$$Ch_{\mathbb{Z}_p[[\Gamma]]}(X(T_p E / \mathbb{Q}_{\text{cyc}})) = Ch_{\mathbb{Z}_p[[\Gamma]]}(X(T_p E^* / \mathbb{Q}_{\text{cyc}})^\iota), \quad (4)$$

as ideals in the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$. (Notice that by Weil pairing, $T_p E^* \cong T_p E$).

Indeed, under certain assumptions, (4) is a corollary of the main conjecture of Iwasawa theory for elliptic curves (proven in [S-U, Corollary 3.34], also see [Ka]) together with the fact that the p -adic L -function g_E satisfies functional equation (3). However, for any elliptic curve E defined over \mathbb{Q} which is ordinary at p , (4) was already proven by [Gr1, Theorem 2] (and also independently in [Pe, Theorem 4.2.1]) purely algebraically and without assuming the existence of g_E . The proofs uses duality and pairing in cohomology, (like the Poitou-Tate duality, generalized Cassels-Tate pairing of Flach) among other tools.

Now for any compatible system of l -adic representations associated to a motive, a complex L -function is defined and one can think of similar questions. For example, for a normalized cuspidal Hilbert eigenform f , which is nearly ordinary at primes \mathfrak{p} of a totally real number field F dividing the prime p , one can associate a compatible system of l -adic representation. Furthermore, one can define a Selmer group $X(T_f / F_{\text{cyc}})$ using the Galois representation of the Galois group $G_F := \text{Gal}(\bar{F}/F)$. Under suitable conditions (for example, non-critical slope), the p -adic L -function for f , which interpolates the complex L -function, exists (see [Di], also see [Sa]); and a precise Iwasawa main conjecture for f over F_{cyc} can also be formulated (for example, see [Wa]). However, this cyclotomic Iwasawa main conjecture for f is not proven yet. In Theorem 2.10, we prove the functional equation for the characteristic ideal of the Selmer group of f i.e.

$$Ch_{O_f[[\Gamma]]}(X(T_f / F_{\text{cyc}})) = Ch_{O_f[[\Gamma]]}(X(T_f^* / F_{\text{cyc}})^\iota), \quad (5)$$

algebraically (without assuming the existence of the p -adic L -function or the Iwasawa main conjecture of f). Thus, Theorem 2.10 can be thought of as a modest evidence towards the validity for the cyclotomic Iwasawa main conjecture for f .

Now, let us consider the nearly ordinary Hida deformation of Hilbert modular forms. A ‘several variable’ p -adic L -function (say \mathcal{L}^p) associated to (a branch \mathcal{R} of the) nearly ordinary Hida family $\mathbf{H}_{\mathcal{N}, O}$ over F_{cyc} will interpolate the special values of the complex L -functions of the various individual nearly ordinary normalized

cuspidal Hilbert eigenforms lying in the family (cf. [Oc4], [Di]). Hence, \mathcal{L}^p should also satisfy a functional equation. Again, by the ‘several variable’ Iwasawa main conjecture over F_{cyc} for a nearly ordinary Hida family $\mathbf{H}_{\mathcal{N},O}$ of Hilbert modular forms (cf. [Wa]), \mathcal{L}_p should be a characteristic ideal of the ‘big’ Selmer group $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})$ of the branch \mathcal{R} of the nearly ordinary Hida family. Thus, we would expect to get a ‘functional equation’ stating

$$Ch_{R[[\Gamma]]}(\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})) = Ch_{R[[\Gamma]]}(\mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/F_{\text{cyc}})^{\iota}) \quad (6)$$

where $\Gamma = \text{Gal}(F_{\text{cyc}}/F)$. Again, we will prove this fact algebraically (without any assumption on the analytic side) in Theorem 3.10 and thus, in turn, this can be thought of as a modest evidence for the validity of the Iwasawa main conjecture for the nearly ordinary Hida deformation of Hilbert modular forms over F_{cyc} .

From an purely algebraic point of view of duality and pairing in cohomology, Theorem 3.10 can also be thought as a direct generalization of result of [Gr1, Theorem 2] in the nearly ordinary Hida deformation setting.

An entirely parallel argument as above, in the setting of ordinary Hida deformation of elliptic modular forms over the \mathbb{Z}_p^2 extension of an imaginary quadratic field K , works as the motivation for Theorem 4.9. (Functional equation for elliptic modular forms over the cyclotomic \mathbb{Z}_p extension of K was discussed in [J-P, Theorem 5.2].) In this case though, we would like to stress that for an imaginary quadratic field K , under certain hypotheses, the three variable Iwasawa main conjecture over K_{∞} for an ordinary Hida family of elliptic modular form has been proven in [S-U, § 3.6.3]. Indeed, in that article a suitable three variable p -adic L -function for the ordinary Hida family has been constructed ([S-U, § 3.4.5]). It is known that this 3 variable p -adic L -function should satisfy a functional equation. Thus, at least in principle, the work of [S-U] should also establish the equality

$$Ch_{\mathcal{R}[[\Gamma_K]]}(\mathcal{X}(T_{\mathcal{R}}/K_{\infty})) = Ch_{\mathcal{R}[[\Gamma_K]]}(\mathcal{X}(T_{\mathcal{R}}^*/K_{\infty})^{\iota})$$

in $\mathcal{R}[[\Gamma_K]]$. However, we would like to mention that our proof of the functional equation for the Selmer group $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/K_{\infty})$ (Theorem 4.9) is simple and we do not need to make use of the vast tools involved in the proof of the 3 variable main conjecture of [S-U]. Moreover, in Theorem 4.9, we do not need some of the hypotheses of the proof of the main conjecture (see [S-U, § 3.6]).

The key idea of the proof of Theorem 2.10 is to use generalized Cassels-Tate pairing of Flach along with a “control theorem” (Theorem 2.3). The central idea of the proof of Theorem 3.10 (and Theorem 4.9) can be explained in three steps. First, we show that for infinitely many arithmetic points the specialization map is a pseudo-isomorphism. Secondly, we use the fact that functional equation holds at the fibre for infinitely many arithmetic specialization. Finally, we use some suitable lifting techniques, generalization of results of [Oc3], to obtain our results. This gives a simple proof of the desired functional equation of the ‘big’ Selmer group of the nearly ordinary Hida family.

The structure of the article is as follows. In section 1, we discuss some preliminary results in two parts. In subsection 1.1, we discuss preliminaries related to the Hida deformation for nearly ordinary Hilbert modular forms and ordinary elliptic modular

forms, only to the extent which we need in this article. In subsection 1.2, we define various Selmer group involved. In section 2, we prove a control theorem for Hilbert modular form and deduce Theorem 2.10. In section 3, we discuss the specialization results connecting the ‘big’ Selmer groups with the Selmer groups of the individual Hilbert modular forms at the fibres and prove the main theorem in the Hilbert modular form case (Theorem 3.10). We prove the second version of the main theorem for elliptic modular forms over \mathbb{Z}_p^2 extension in section 4 (Theorem 4.9).

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1. PRELIMINARIES

1.1. Preliminary results on Hilbert Modular Forms.

1.1.1. *Hilbert Modular Forms and Nearly Ordinary Hecke Algebra.* In this subsection, we collect some basic results about nearly ordinary Hida deformation of Hilbert modular forms and ordinary Hida deformation of elliptic modular forms, which are needed in the course of this article. All the results in this section are well known and can be found in the literature (cf. [Hi1], [Hi2], [Wi1], [Wi2]). Our presentation of results in this subsection, in many cases, follows the presentation of [F-O].

Let p be an odd prime. Let F be a totally real number field of degree d , \mathcal{O}_F be the ring of integers of F , and J_F denotes the set of embedding of F into \mathbb{R} . To an ideal \mathcal{M} of \mathcal{O}_F , we attach standard compact open subgroups K_0, K_1 and K_{11} of $Gl_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})$ as follows:

$$\begin{aligned} K_0(\mathcal{M}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}) \mid c \equiv 0 \pmod{\mathcal{M}} \right\} \\ K_1(\mathcal{M}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathcal{M}) \mid d \equiv 1 \pmod{\mathcal{M}} \right\} \\ K_{11}(\mathcal{M}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathcal{M}) \mid a, d \equiv 1 \pmod{\mathcal{M}} \right\}. \end{aligned}$$

Definition 1.1. A weight $k = \sum_{\tau \in J_F} k_{\tau} \tau$ is an element of $\mathbb{Z}[J_F]$, an arithmetic weight is a weight such that $k_{\tau} \geq 2$ for all $\tau \in J_F$ and k_{τ} has constant parity. A parallel weight is an integral multiple of the weight $t = \sum_{\tau \in J_F} \tau$. Two weights are said to be equivalent if their deference is a parallel weight. To an arithmetic weight k , one associates a weight $v \in \mathbb{Z}[J_F]$, called the parallel defect of k , which satisfies $k + 2v \in \mathbb{Z}t$.

Let \mathcal{O} be the ring of integers of a finite extension of \mathbb{Q}_p which contains all conjugates of F . For k an arithmetic weight and v its parallel defect, $S_{k,w}(U; \mathcal{O})$ denotes the holomorphic cusp forms of weight (k, w) of level U and coefficient in \mathcal{O} , where U is a finite index subgroup of $Gl_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})$ containing $K_{11}(\mathcal{M})$ for some $\mathcal{M} \subset \mathcal{O}_F$,

and $w = k + v - t$. A cuspidal Hilbert modular form $S_{k,w}(U; \mathcal{O})$ is called primitive if it is not a Hilbert modular form of weight (k, w) and of level smaller than U . A normalized primitive Hilbert eigenform is called a Hilbert newform. To $f \in S_{k,w}(U; \mathcal{O})$, one can naturally associate an automorphic representation $\pi_f \in Gl_2(\mathbb{A}_F)$.

Fix an ideal \mathcal{N} of F which is prime to p and for any $s \in \mathbb{N}$, we have an action of $G = (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^* \times ((\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^* / \mathcal{O}_F^*)$ on $S_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$. We have an action of the p -Hecke operator $T_0(p)$, normalized according to the parallel defect v , on the space $S_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$. The largest \mathcal{O} submodule of $S_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$ on which $T_0(p)$ acts invertibly is denoted by $S_{k,w}^{n.o.}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$. A form $f \in S_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$ is called nearly ordinary if $f \in S_{k,w}^{n.o.}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$.

The nearly ordinary Hecke algebra $\mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$ of weight (k, w) and level $K_1(\mathcal{N}) \cap K_{11}(p^s)$ is defined to be the \mathcal{O} subalgebra of $End_{\mathcal{O}}(S_{k,w}^{n.o.}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O}))$ generated by the Hecke operators. The \mathcal{O} algebra $\mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$ is finite flat over \mathcal{O} .

Let k be an arithmetic weight and v be its parallel defect. By the perfect duality between $\mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$ and $S_{k,w}^{n.o.}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$, giving an eigen cuspform $f \in S_{k,w}^{n.o.}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$ is equivalent to giving an algebra homomorphism

$$q_f : \mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O}) \rightarrow \mathbf{H}_{k,w}(K_1(\mathcal{N}p^s); \mathcal{O}) \rightarrow \bar{\mathbb{Q}}_p$$

sending $T \in \mathbf{H}_{k,w}(K_1(\mathcal{N}p^s); \mathcal{O})$ to $a_1(f|T)$.

Let $\Lambda_{\mathcal{O}}$ denote the completed group algebra $\mathcal{O}[G/G_{tors}]$. The algebra $\Lambda_{\mathcal{O}}$ is non-canonically isomorphic to the power series algebra $\mathcal{O}[[X_1, \dots, X_r]]$, where $r = 1 + d + \delta_{F,p}$, $\delta_{F,p}$ be the defect of the Leopoldt's conjecture for F at p .

Let the nearly ordinary Hecke algebra $\mathbf{H}_{\mathcal{N}, \mathcal{O}}$ be the inverse limit w.r.t. s of the $\mathbf{H}_{2t,0}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$. By fundamental work of Hida, for any arithmetic weight $k \in \mathbb{Z}[J_F]$, we have $\mathbf{H}_{\mathcal{N}, \mathcal{O}} \cong \varprojlim_s \mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$. The nearly ordinary

Hecke algebra $\mathbf{H}_{\mathcal{N}, \mathcal{O}}$ is finite torsion free $\Lambda_{\mathcal{O}}$ module and hence a semi-local ring. Let \mathfrak{a} be one of the finitely many ideals of height zero in $\mathbf{H}_{\mathcal{N}, \mathcal{O}}$. The algebra $\mathcal{R} = \mathbf{H}_{\mathcal{N}, \mathcal{O}} / \mathfrak{a}$ is called a branch of $\mathbf{H}_{\mathcal{N}, \mathcal{O}}$.

Definition 1.2. For a weight $k \in \mathbb{Z}[J_F]$, an algebraic character $\xi : G = (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^* \times ((\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^* / \mathcal{O}_F^*) \rightarrow \bar{\mathbb{Q}}_p^*$ of weight (k, w) is a character of the form, $\xi(a, z) = \psi(a, z) \chi_{\text{cyc}}^{[n+2v]}(z) a^n$, where ψ is a character of finite order, $[n+2v]$ is the unique integer satisfying $n+2v = [n+2v]t$ (Recall, $n = k + 2t$, $w = k + v - t$). An algebraic character of weight (k, w) is an arithmetic character of weight (k, w) if its restriction to $\mathcal{O}_F^* \subset (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^*$ is trivial. An algebra homomorphism, $\xi \in \text{Hom}(\Lambda_{\mathcal{O}}, \bar{\mathbb{Q}}_p)$ is algebraic (resp. arithmetic) of weight (k, w) if $\xi|_G$ is algebraic (resp. arithmetic) of weight (k, w) . If R is a finite $\Lambda_{\mathcal{O}}$ algebra, $\xi \in \text{Hom}(R, \bar{\mathbb{Q}}_p)$ is algebraic (resp. arithmetic) of weight (k, w) if $\xi|_{\Lambda_{\mathcal{O}}}$ is algebraic (resp. arithmetic) of weight (k, w) . A prime ideal $P_{\xi} \subset R$ which is defined as the kernel of an algebraic (resp. arithmetic) specialization of ξ of R is called an algebraic (resp. arithmetic) point.

For any k be an arithmetic weight and any of its parallel defect v , and for any nearly ordinary eigen cuspform $f \in S_{k,w}^{n.o.}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$ which is new at every prime dividing \mathcal{N} , there exists a unique branch \mathcal{R} of $\mathbf{H}_{\mathcal{N}, \mathcal{O}}$ and a unique arithmetic specialization $\xi_f : \mathcal{R} \rightarrow \bar{\mathbb{Q}}_p$ of weight (k, w) such that $\xi_f(\mathcal{R})$ is canonically identified with $q_f(\mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O}))$.

Let \mathcal{R} be a branch of $\mathbf{H}_{\mathcal{N}, \mathcal{O}}$. Then for any k be an arithmetic weight and any of its parallel defect v , and for any arithmetic specialization $\xi : \mathcal{R} \rightarrow \bar{\mathbb{Q}}_p$ of weight (k, w) , there exists a unique nearly ordinary eigen cuspform $f_\xi \in \mathbf{H}_{k,w}(K_1(\mathcal{N}p^s); \mathcal{O})$ for some s , such that $\xi(\mathcal{R})$ is canonically identified with $q_{f_\xi}(\mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O}))$. We will denote the set of arithmetic points of \mathcal{R} by $\mathfrak{X}(\mathcal{R})$. For each $\xi \in \mathfrak{X}(\mathcal{R})$, $P_\xi = \ker(\xi)$ is a coheight 1 prime ideal in \mathcal{R} .

1.1.2. Galois Representation. Galois representation associated to a Hilbert modular eigen cuspform was constructed and studied by Carayol, Ohta, Wiles, Taylor and Blasius-Rogawski. We briefly recall their results in the following two theorems.

Theorem 1.3. *Let $f \in S_{k,w}(K_1(\mathcal{M}); \bar{\mathbb{Q}}_p)$ be a normalized eigen cuspform of arithmetic weight k , and let K be a finite extension of \mathbb{Q}_p containing all Hecke eigenvalues for f . Then there exists a continuous irreducible G_F representation $V_f \cong K^{\oplus 2}$, which is unramified outside $\mathcal{M}p$ and satisfies*

$$\det(1 - \text{Fr}_\lambda X | V_f) = 1 - T_\lambda(f)X + S_\lambda(f)X^2$$

for all $\lambda \nmid \mathcal{M}p$, where T_λ (resp. S_λ) is the Hecke operator induced by the coset class $K_1(\mathcal{M}) \begin{pmatrix} 1 & 0 \\ 0 & \pi_\lambda \end{pmatrix} K_1(\mathcal{M})$ (resp. $K_1(\mathcal{M}) \begin{pmatrix} \pi_\lambda & 0 \\ 0 & \pi_\lambda \end{pmatrix} K_1(\mathcal{M})$), where π_λ is a uniformizer at λ and Fr_λ is the geometric Frobenius at λ .

The G_F representation V_f is known to be irreducible, and hence characterized upto isomorphism by the above equation.

Remark 1.4. *Let $f \in S_{k,w}(K_1(\mathcal{M}); \bar{\mathbb{Q}}_p)$ be a normalized eigen cuspform of arithmetic weight k , and let K be a finite extension of \mathbb{Q}_p containing all Hecke eigenvalues for f as in Theorem 1.3. If $\mathfrak{p} \mid p$, let $c(\mathfrak{p}, f)$ be the $T(\mathfrak{p})$ eigenvalue of f . We say f is ordinary at \mathfrak{p} if $c(\mathfrak{p}, f)$ is a unit in the ring of integers of K and f is ordinary at p if and only if for all $\mathfrak{p} \mid p$, f is ordinary at \mathfrak{p} .*

Next theorem describes the local properties of the Galois representation V_f .

Theorem 1.5. *Let $f \in S_{k,w}(K_1(\mathcal{M}); \bar{\mathbb{Q}}_p)$ be a normalized eigen cuspform. Let $w_{\max} = \max \{w_\tau, \tau \in J_F\}$. Let V_f (resp. π_f) be the Galois representation (resp. automorphic representation) associated to f .*

(1) *If $\lambda \nmid p$, then*

(a) *The inertia group I_λ at λ acts on V_f through infinite quotient iff $\pi_{f,\lambda}$ is a Steinberg representation. In this case, V_f has a unique filtration by graded pieces of dimension one:*

$$0 \rightarrow (V_f)_\lambda^+ \rightarrow V_f \rightarrow (V_f)_\lambda^- \rightarrow 0$$

which is stable under the decomposition group G_λ at λ . The inertia group I_λ acts on $(V_f)_\lambda^+$ (resp. $(V_f)_\lambda^-$) through a finite quotient of I_λ .

An eigenvalue α of the action of a lift of Fr_λ to G_λ on $(V_f)_\lambda^+$ (resp. $(V_f)_\lambda^-$) is an algebraic number satisfying $|\alpha|_\infty = (N_{F/\mathbb{Q}}(\lambda))^{\frac{w_{max}+1}{2}}$ (resp. $(N_{F/\mathbb{Q}}(\lambda))^{\frac{w_{max}-1}{2}}$).

- (b) If I_λ acts on V_f through a finite quotient, the action of I_λ is reducible iff $\pi_{f,\lambda}$ is principal series. If I_λ acts on V_f through a finite quotient, an eigenvalue α of the action of a lift of Fr_λ to G_λ on V_f is an algebraic number satisfying $|\alpha|_\infty = (N_{F/\mathbb{Q}}(\lambda))^{\frac{w_{max}}{2}}$.
- (2) If $\mathfrak{p}|p$, and if f is nearly ordinary at \mathfrak{p} . Then V_f has a unique filtration by graded pieces of dimension one:

$$0 \rightarrow (V_f)_\mathfrak{p}^+ \rightarrow V_f \rightarrow (V_f)_\mathfrak{p}^- \rightarrow 0$$

which is stable under the decomposition group $G_\mathfrak{p}$ at \mathfrak{p} , and Hodge-Tate weight of $(V_f)_\mathfrak{p}^+$ is greater than Hodge-Tate weight of $(V_f)_\mathfrak{p}^-$.

Remark 1.6. Let $f \in S_{k,w}(K_1(\mathcal{M}); \bar{\mathbb{Q}}_p)$ be a normalized eigen cuspform of arithmetic weight k with associated Galois representation V_f over K as in Theorem 1.3. If f is p -ordinary, then for all primes $\mathfrak{p} | p$,

$$V_f|_{G_\mathfrak{p}} \sim \begin{bmatrix} \epsilon_\mathfrak{p} & * \\ 0 & \delta_\mathfrak{p} \end{bmatrix},$$

where $\epsilon_\mathfrak{p}, \delta_\mathfrak{p}$ are characters of $G_\mathfrak{p}$ with values in K^* and $\delta_\mathfrak{p}$ is unramified. In the case of nearly ordinary f at \mathfrak{p} , our Galois representation restricted $G_\mathfrak{p}$ looks same, except $\delta_\mathfrak{p}$ need not be unramified.

The following two theorems are the Hida family versions of the two above theorem, and are due to the work of Wiles and Hida.

Theorem 1.7. Let \mathcal{R} be a branch of $\mathbf{H}_{N,\mathcal{O}}$. Then there exists a finitely generated torsion free \mathcal{R} module $\mathcal{T}_\mathcal{R}$ with continuous G_F action, which satisfies the following properties:

- (1) The vector space $\mathcal{V}_\mathcal{R} := \mathcal{T}_\mathcal{R} \otimes_\mathcal{R} \mathcal{K}$ is of dimension 2 over \mathcal{K} , where \mathcal{K} is the fraction field of \mathcal{R} .
- (2) The representation $\rho_\mathcal{R}$ of G_F on $\mathcal{V}_\mathcal{R}$ is irreducible and unramified outside primes dividing $Np\infty$.
- (3) For any arithmetic weight (k, w) , and for any nearly ordinary eigen cuspform $f \in S_{k,w}^{n.o.}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$ which corresponds to the arithmetic weight $\xi = \xi_f$ on the branch \mathcal{R} , $T_f = \mathcal{T}_\mathcal{R} \otimes_\mathcal{R} \xi_f(\mathcal{R})$ is a lattice of the Galois representation V_f associated to f .

Next theorem characterizes the local behavior of the Galois representation associated to \mathcal{R} .

Theorem 1.8. Let \mathcal{R} be a branch of $\mathbf{H}_{N,\mathcal{O}}$ and $\mathcal{V}_\mathcal{R} = \mathcal{V}$ be the Galois representation over \mathcal{K} . Then,

- (1) For every prime $\lambda \nmid Np$, we have:

$$\det(1 - Fr_\lambda X | \mathcal{V}) = 1 - T_\lambda X + S_\lambda X^2$$

where T_λ and S_λ are Hecke operators on \mathcal{R} at λ which is obtained at the limit of the Hecke operators in Theorem 1.3 at finite levels.

- (2) For every prime $\mathfrak{p}|p$, we have a canonical filtration obtained as the limit of the filtration given in Theorem 1.5:

$$0 \rightarrow \mathcal{V}_{\mathfrak{p}}^+ \rightarrow \mathcal{V} \rightarrow \mathcal{V}_{\mathfrak{p}}^- \rightarrow 0$$

which is stable under the action of the decomposition group $G_{\mathfrak{p}}$ at \mathfrak{p} .

For an ideal \mathcal{N} in \mathcal{O}_F which is prime to p , we have a Hida's nearly ordinary Hecke algebra $\mathbf{H}_{\mathcal{N}, \mathcal{O}}$. We fix a branch \mathcal{R} of $\mathbf{H}_{\mathcal{N}, \mathcal{O}}$ and a representation $\mathcal{T} = \mathcal{T}_{\mathcal{R}}$ as in Theorem 1.7. We assume that we have a $G_{\mathfrak{p}}$ stable filtration

$$0 \rightarrow \mathcal{T}_{\mathfrak{p}}^+ \rightarrow \mathcal{T} \rightarrow \mathcal{T}_{\mathfrak{p}}^- \rightarrow 0$$

by finite type \mathcal{R} modules $\mathcal{T}_{\mathfrak{p}}^+$ and $\mathcal{T}_{\mathfrak{p}}^-$ with continuous $G_{\mathfrak{p}}$ action which gives the exact sequence

$$0 \rightarrow \mathcal{V}_{\mathfrak{p}}^+ \rightarrow \mathcal{V} \rightarrow \mathcal{V}_{\mathfrak{p}}^- \rightarrow 0$$

in Theorem 1.8 by taking base extension to \mathcal{K} .

Let $\mathfrak{m}_{\mathcal{R}}$ denote the maximal ideal of \mathcal{R} and $\mathbb{F}_{\mathcal{R}}$ be the finite field $\mathcal{R}/\mathfrak{m}_{\mathcal{R}}$. Associated to the Galois representation of G_F in Theorem 1.7, there exist a canonical residual Galois representation $\bar{\rho}_{\mathcal{R}} : G_F \rightarrow GL_2(\mathbb{F}_{\mathcal{R}})$. Throughout this article, we assume the following two hypotheses on this $\bar{\rho}$.

Hypothesis 1.9. (Irr): *The residual representation $\bar{\rho}_{\mathcal{R}}$ of G_F is absolutely irreducible.*

Hypothesis 1.10. (Dist): *As before, let $G_{\mathfrak{p}}$ be the decomposition subgroup of G_F at \mathfrak{p} . The restriction of the residual representation at the decomposition subgroup i.e. $\bar{\rho}_{\mathcal{R}}|_{G_{\mathfrak{p}}}$ is an extension of two distinct characters of $G_{\mathfrak{p}}$ with values in $\mathbb{F}_{\mathcal{R}}^*$ for each $\mathfrak{p}|p$.*

Remark 1.11. *Conditions (Irr) and (Dist) together implies that the representation \mathcal{T} can be chosen to be free of rank two over \mathcal{R} and, for each $\mathfrak{p} | p$, the graded pieces $\mathcal{T}_{\mathfrak{p}}^+$ and $\mathcal{T}_{\mathfrak{p}}^-$ are both free of rank one over \mathcal{R} .*

1.1.3. *ordinary Hida deformation of elliptic modular forms.* We summarize the relevant results for p -ordinary Hida family of elliptic modular forms in the following remark. This details can be found in the literature (also in [J-P, Section 1]).

Remark 1.12. *Let $f = \sum a_n q^n$ be a normalized elliptic eigenform of weight $k \geq 2$, level N and nebentypus ψ . We say that f is p -ordinary if $\iota_p(a_p)$ is a p -adic unit. Also assume that f is p -stabilized (p -refined). By the work of Eichler-Shimura, Deligne, Mazur-Wiles, Wiles and many other people, to such an f , one can associate a Galois representation $\rho_f : G_{\mathbb{Q}} \rightarrow GL(V_f)$, here V_f is a two dimensional vector space over a finite extension of \mathbb{Q}_p , which is unramified outside Np , and for any prime $l \nmid Np$, arithmetic Frob_l has the characteristic polynomial $X^2 - a_l X + \psi(l)l^{k-1}$, moreover, restricted to the decomposition subgroup at p , we have, $V_f|_{G_p} \sim \begin{bmatrix} \epsilon_1 & * \\ 0 & \epsilon_2 \end{bmatrix}$ with ϵ_2 unramified. We have a notion of Hida family and arithmetic points for elliptic*

modular forms of tame level N . A theorem of Hida asserts that if $(N, p) = 1$ and f is a p -stabilized newform of weight $k \geq 2$, tame level N , then there exists a branch \mathcal{R} of an ordinary Hida family and an arithmetic point $\xi : \mathcal{R} \rightarrow \mathbb{Q}_p$ such that $\xi(\mathcal{R})$ canonically corresponds with f . By the work of Hida and Wiles, to \mathcal{R} one can associate a big Galois representation of dimension two, $\rho_{\mathcal{R}} : G_{\mathbb{Q}} \rightarrow GL(\mathcal{V}_{\mathcal{R}})$, where $\mathcal{V}_{\mathcal{R}}$ is a vector space of dimension two over K , the fraction field of \mathcal{R} . The Galois representation $\rho_{\mathcal{R}}$ is unramified at all finite places outside Np and for a prime $l \nmid Np$, one has $\det(1 - Fr_l X|_{\mathcal{V}}) = 1 - T_l X + S_l X^2$, where T_l, S_l are Hecke operators on \mathcal{R} at l , moreover, $\mathcal{V}_{\mathcal{R}}|_{G_p} \sim \begin{bmatrix} \tilde{\epsilon}_1 & * \\ 0 & \tilde{\epsilon}_2 \end{bmatrix}$ with $\tilde{\epsilon}_2$ unramified. Similar to the case of nearly ordinary Hilbert modular forms, throughout the article, we make the following two hypotheses.

Hypothesis 1.13. (Irr): The residual representation $\bar{\rho}_{\mathcal{R}}$ of $G_{\mathbb{Q}}$ is absolutely irreducible.

Hypothesis 1.14. (Dist): For the representation $\bar{\rho}_{\mathcal{R}}|_{G_p}$, we have $\tilde{\epsilon}_1 \neq \tilde{\epsilon}_2 \pmod{\mathfrak{m}_{\mathcal{R}}}$.

Under **(Irr)** and **(Dist)**, we have the lattices T_f in V_f and $\mathcal{T}_{\mathcal{R}}$ in $\mathcal{V}_{\mathcal{R}}$ invariant under ρ_f and $\rho_{\mathcal{R}}$ respectively, such that $T_f \cong \mathcal{T}_{\mathcal{R}} \otimes_{\mathcal{R}} \xi(\mathcal{R})$. Moreover, $\mathcal{T}_{\mathcal{R}}$ has a filtration as G_p module,

$$0 \rightarrow \mathcal{T}_{\mathcal{R}}^+ \rightarrow \mathcal{T}_{\mathcal{R}} \rightarrow \mathcal{T}_{\mathcal{R}}^- \rightarrow 0$$

such that the graded pieces $\mathcal{T}_{\mathcal{R}}^+$ and $\mathcal{T}_{\mathcal{R}}^-$ are free \mathcal{R} module of rank one.

1.2. Various Selmer groups. We fix a totally real number field F and as $S = S_F$ is a finite set of finite places of F containing the primes lying above $\mathcal{N}p$. Let $f \in S_{k,w}^{n.o.}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$. Let V_f denote the Galois representation associated to f over K_f , a finite extension of \mathbb{Q}_p containing all Hecke eigenvalues of f as in Theorem 1.3. We denote the ring of integer of K_f by O_f . Let $T_f \subset V_f$ be a G_F invariant lattice. Thus we have an induced action of G_F on the discrete module $A_f := V_f/T_f$. Further, since ρ_f is nearly ordinary at p , A_f has a filtration as a G_p module for every $\mathfrak{p}|p$,

$$0 \rightarrow (A_f)_{\mathfrak{p}}^+ \rightarrow A_f \rightarrow (A_f)_{\mathfrak{p}}^- \rightarrow 0, \quad (7)$$

where both $(A_f)_{\mathfrak{p}}^{+\vee}$ and $(A_f)_{\mathfrak{p}}^{-\vee}$ are free over O_f of rank 1.

Let \mathcal{L} be an finite or infinite algebraic extension of F and w will denote a prime in \mathcal{L} . The notation $w | S$ will mean the prime w of \mathcal{L} is lying above a prime in S . Given such a prime w , let G_w and I_w respectively denote the decomposition subgroup and inertia subgroup for the extension $\bar{\mathbb{Q}}/\mathcal{L}$ with respect to the primes \bar{w}/w , where we have fixed a prime \bar{w} of $\bar{\mathbb{Q}}$ over w . We denote the Frobenius element at w by Fr_w so that $G_w/I_w \cong \langle \text{Fr}_w \rangle$.

Let \mathcal{R} be a branch of $\mathbf{H}_{\mathcal{N}, \mathcal{O}}$. By Theorem 1.7, we have a free \mathcal{R} lattice $\mathcal{T} = \mathcal{T}_{\mathcal{R}}$. Define,

$$\mathcal{A} = \mathcal{A}_{\mathcal{R}} := \mathcal{T} \otimes_{\mathcal{R}} \text{Hom}_{\text{cont}}(\mathcal{R}, \mathbb{Q}_p/\mathbb{Z}_p).$$

For any arithmetic character ξ of \mathcal{R} , we have from definition 1.2, $\mathcal{A}[P_{\xi}] \cong A_{f_{\xi}}$. Using Theorem 1.8 and by the assumption 1.10, we get \mathcal{A} has a filtration as a G_p module

$$0 \rightarrow \mathcal{A}_{\mathfrak{p}}^+ \rightarrow \mathcal{A} \rightarrow \mathcal{A}_{\mathfrak{p}}^- \rightarrow 0, \quad (8)$$

where both $\mathcal{A}_{\mathfrak{p}}^{+\vee}$ and $\mathcal{A}_{\mathfrak{p}}^{-\vee}$ are free \mathcal{R} modules of rank 1. Also we have $\mathcal{A}_{\mathfrak{p}}^{-}[P_{\xi}] \cong (A_{f_{\xi}})_{\mathfrak{p}}^{-}$.

Let L be a finite extension of K . We define various Selmer groups associated to f and \mathcal{R} , defined over L .

Definition 1.15 (Greenberg Selmer group of f).

$$S(A_f/L) = \ker(H^1(F_S/L, A_f) \longrightarrow \bigoplus_{w|S, w \nmid p} H^1(I_w, A_f)^{G_w} \oplus \bigoplus_{w|\mathfrak{p}|p} H^1(I_w, (A_f)_{\mathfrak{p}}^{-})^{G_w}) \quad (9)$$

Definition 1.16 (Strict Selmer group of f).

$$S'(A_f/L) := \ker(H^1(F_S/L, A_f) \longrightarrow \bigoplus_{w|S, w \nmid p} H^1(I_w, A_f)^{G_w} \oplus \bigoplus_{w|\mathfrak{p}|p} H^1(G_w, (A_f)_{\mathfrak{p}}^{-})) \quad (10)$$

Definition 1.17 (Greenberg Selmer group of \mathcal{R}).

$$S(\mathcal{A}/L) = \ker(H^1(F_S/L, \mathcal{A}) \longrightarrow \bigoplus_{w|S, w \nmid p} H^1(I_w, \mathcal{A})^{G_w} \oplus \bigoplus_{w|\mathfrak{p}|p} H^1(I_w, \mathcal{A}_{\mathfrak{p}}^{-})^{G_w}) \quad (11)$$

Definition 1.18 (Strict Selmer group of \mathcal{R}).

$$S'(\mathcal{A}/L) := \ker(H^1(F_S/L, \mathcal{A}) \longrightarrow \bigoplus_{w|S, w \nmid p} H^1(I_w, \mathcal{A})^{G_w} \oplus \bigoplus_{w|\mathfrak{p}|p} H^1(G_w, \mathcal{A}_{\mathfrak{p}}^{-})) \quad (12)$$

Definition 1.19. Let $S^{\perp} \in \{S, S'\}$. Define the Pontryagin dual

$$X^{\perp}(T_f/L) = \text{Hom}_{\text{cont}}(S^{\perp}(A_f/L), \mathbb{Q}_p/\mathbb{Z}_p) \quad (13)$$

where $X^{\perp} = X$ if $S^{\perp} = S$ and $X^{\perp} = X'$ if $S^{\perp} = S'$.

Define $\mathcal{X}^{\perp}(\mathcal{T}_{\mathcal{R}}/L)$ by replacing X^{\perp} by \mathcal{X}^{\perp} , A_f by \mathcal{A} and T_f with $\mathcal{T}_{\mathcal{R}}$ in (13).

For an infinite extension F_{∞} of F , the Selmer group $S^{\perp}(A_f/F_{\infty})$ (respectively $S^{\perp}(\mathcal{A}/F_{\infty})$) is defined by taking the inductive limit of $S^{\perp}(A_f/L')$ (resp. $S^{\perp}(\mathcal{A}/L')$) over all finite extensions L' of F contained in F_{∞} with respect to the natural restriction maps. The corresponding Pontryagin duals are denoted by $X(T_f/F_{\infty})$ and $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\infty})$ respectively.

Under the natural action of $\Gamma = \text{Gal}(F_{\text{cyc}}/F)$, $X(T_f/F_{\text{cyc}})$ (respectively $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})$) acquires the structure of a $O_f[[\Gamma]]$ (respectively $\mathcal{R}[[\Gamma]]$) module. Also note that for $B \in \{A_f, \mathcal{A}\}$, we have $0 \longrightarrow S'(B/F_{\text{cyc}}) \hookrightarrow S(B/F_{\text{cyc}})$.

Next we discuss various types of twisted Selmer groups. For any \mathbb{Z}_p module M , let $M(1)$ denotes the Tate twist of M by the p -adic cyclotomic character $\chi_p : \Gamma \longrightarrow \mathbb{Z}_p^{\times}$. Define

$$\mathcal{T}_{\mathcal{R}}^* := \text{Hom}_{\mathcal{R}}(\mathcal{T}_{\mathcal{R}}, \mathcal{R}(1)).$$

We have a corresponding filtration of \mathcal{T}^* as a $G_{\mathfrak{p}}$ module

$$0 \longrightarrow (\mathcal{T}_{\mathcal{R}}^*)_{\mathfrak{p}}^+ \longrightarrow \mathcal{T}_{\mathcal{R}}^* \longrightarrow (\mathcal{T}_{\mathcal{R}}^*)_{\mathfrak{p}}^- \longrightarrow 0,$$

where the graded pieces are defined as $(\mathcal{T}_{\mathcal{R}}^*)_{\mathfrak{p}}^+ := \text{Hom}_{\mathcal{R}}((\mathcal{T}_{\mathcal{R}})_{\mathfrak{p}}^-, \mathcal{R}(1))$ and $(\mathcal{T}_{\mathcal{R}}^*)_{\mathfrak{p}}^- := \text{Hom}_{\mathcal{R}}((\mathcal{T}_{\mathcal{R}})_{\mathfrak{p}}^+, \mathcal{R}(1))$. We can now define

$$\mathcal{A}^* := \mathcal{T}_{\mathcal{R}}^* \otimes_{\mathcal{R}} \text{Hom}_{\text{cont}}(\mathcal{R}, \mathbb{Q}_p/\mathbb{Z}_p).$$

From the above discussion, we can get a filtration of \mathcal{A}^* as in (8).

Also, corresponding to a newform f , we define

$$T_f^* := \text{Hom}_{O_f}(T_f, O_f(1)).$$

Then it is easy to see that the quotient $\mathcal{T}^* \otimes_{\mathcal{R}} \xi(\mathcal{R})$ is isomorphic to $T_{f_\xi}^*$. Also define $A_f^* = T_f^* \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Then as in (7), there is a filtration

$$0 \longrightarrow (A_f^*)_{\mathfrak{p}}^+ \longrightarrow A_f^* \longrightarrow (A_f^*)_{\mathfrak{p}}^- \longrightarrow 0,$$

with $(A_f^*)_{\mathfrak{p}}^{+\vee}$ and $(A_f^*)_{\mathfrak{p}}^{-\vee}$ are free O_f module of rank 1.

From the above discussions, by making obvious modifications in the definitions 1.15, 1.17 and 1.19, we can now define the Greenberg Selmer groups $S(A_f^*/\mathcal{L})$, $S(\mathcal{A}^*/\mathcal{L})$ and their respective Pontryagin duals $X(T_f^*/\mathcal{L})$ and $\mathcal{X}(T_{\mathcal{R}}^*/\mathcal{L})$ for any finite or infinite extension \mathcal{L} of F .

Let $\rho : \Gamma \longrightarrow O_f^\times$ be a character. Set $T_\rho = O_f(\rho)$, the G_F module with underlying group O_f and an G_F action on it via ρ . Set $T_f(\rho) = T_\rho \otimes_{O_f} T_f$, $A_f(\rho) = T_\rho \otimes_{O_f} A_f$ and $(A_f)_\rho^-(\rho) = T_\rho \otimes_{O_f} (A_f)_\rho^-$ with the diagonal action of G_F . Let M be an $O_f[[\Gamma]]$ module. Define $M(\rho) = T_\rho \otimes_{O_f} M$ with $\gamma \in \Gamma$ acting by diagonal action. Applying $\otimes_{O_f} T_\rho$ to the filtration in (7), we get a filtration for $T_\rho \otimes A_f$. Proceeding in a way similar to the definition 1.15, define the Greenberg Selmer groups with respect to the ‘twist’ ρ , $S(A_f \otimes T_\rho/\mathcal{L})$ and $X(T_\rho \otimes T_f/\mathcal{L})$, for any extension \mathcal{L} of F (possibly infinite). As ρ acts trivially on $G_{F_{\text{cyc}}}$, we notice that

$$X(T_\rho \otimes T_f/F_{\text{cyc}}) \cong X(T_f/F_{\text{cyc}}) \otimes T_{\rho^{-1}}. \quad (14)$$

In particular $X(T_\rho \otimes T_f/F_{\text{cyc}})$ is a finitely generated torsion $O_f[[\Gamma]]$ module if and only if $X(T_f/F_{\text{cyc}})$ is so.

Remark 1.20. Let f be a newform nearly ordinary at p . Then we can express $T_f^* \cong T_{f^*} \otimes O_f(\chi_p^t)$ as Galois modules for some $t \in \mathbb{Z}$ and for some newform f^* which is nearly ordinary at p , has the same level and weight as f but possibly different character. Hence for any extension \mathcal{L} of F , we deduce that $X(T_\rho \otimes T_f^*/\mathcal{L}) \cong X(T_{\rho'} \otimes T_{f^*}/\mathcal{L})$ for certain character $\rho' : \Gamma \longrightarrow O_f^\times$. In particular, $X(T_f^*/F_{\text{cyc}}) \cong X(T_{f^*}/F_{\text{cyc}}) \otimes O_f(s)$ for some $s \in \mathbb{Z}$.

Remark 1.21. Selmer group for p -ordinary elliptic modular forms and for the corresponding Hida family:

Let $f \in S_k(\Gamma_0(Np^r), \psi)$ be a p -ordinary, p -stabilized (elliptic) newform. Then by remark 1.12, via its Galois representation, we can associate to f , a lattice T_f . Set $A_f = V_f/T_f$. Assume the conditions (Irr) and (Dist). Then, again by remark 1.12, to a branch \mathcal{R} of a p -ordinary Hida family of elliptic modular forms of tame level N , we have a free \mathcal{R} lattice $\mathcal{T}_{\mathcal{R}}$ and we can define $\mathcal{A} = \mathcal{A}_{\mathcal{R}} := \mathcal{T} \otimes_{\mathcal{R}} \text{Hom}_{\text{cont}}(\mathcal{R}, \mathbb{Q}_p/\mathbb{Z}_p)$. Then by p -ordinarity, both A_f and \mathcal{A} are equipped with canonical filtration as G_p modules.

Let K be an imaginary quadratic field and K_{cyc} and K_∞ be respectively, the unique cyclotomic and \mathbb{Z}_p^2 extension of K . We assume that p splits in K and the discriminant D_K is coprime to N . Let $S = S_K$ be a finite set of places of K containing the primes dividing Np . Then proceeding in a similar way as in definitions 1.15 - 1.19, we can define the Greenberg Selmer groups and the strict Selmer group of f and \mathcal{R}

over K_{cyc} and K_{∞} . In fact, we will use the same symbols used in the above definitions. However, there is no case of confusion, as we deal with the elliptic modular forms and their ordinary Hida family only in section 4.

2. FUNCTIONAL EQUATION FOR A HILBERT MODULAR FORM

A control theorem is a widely used tool in Iwasawa theory. We prove a ‘control theorem’ for the twisted Selmer group $X(T_f \otimes T_{\rho}/F_{\text{cyc}})$ with ρ as before. A control theorem in case of elliptic modular forms was discussed in [J-P, Theorem 3.1]. Recall, there is a tower of fields $F = F_0 \subset \dots \subset F_n \subset \dots \subset F_{\text{cyc}}$ are such that $\text{Gal}(F_n/F) \cong \mathbb{Z}/p^n\mathbb{Z}$. Set $\Gamma_n = \text{Gal}(F_{\text{cyc}}/F_n)$. Given a cuspidal Hilbert newform $f \in S_{k,w}^{n,o}(K_1(N) \cap K_{11}(p^s); \mathcal{O})$, let ρ_f and π_f respectively be the associated Galois representation and automorphic representation. Let f be nearly ordinary at every prime \mathfrak{p} in F , $\mathfrak{p} \mid p$. Then for any such \mathfrak{p} dividing p , the Frobenius $\text{Fr}_{\mathfrak{p}}$ acts on the 1 dimensional subspace $V_{f,\mathfrak{p}}^-$ with the eigenvalue $\alpha_{f,\mathfrak{p}}$ (say).

Definition 2.1. Let f be as above. Define f to be exceptional if $|\alpha_{f,\mathfrak{p}}|_{\mathbb{C}} = 1$ for some $\mathfrak{p} \mid p$.

Remark 2.2. From the local Langlands correspondence for Hilbert modular forms due to Carayol [Ca] and generalized Ramanujan Conjecture, which is known for Hilbert modular form due to Blassias [Bl]; it follows that the condition of f being exceptional i.e. $|\alpha_{f,\mathfrak{p}}|_{\mathbb{C}} = 1$ for some $\mathfrak{p} \mid p$ happens if for some $\mathfrak{p} \mid p$, the \mathfrak{p} component $\pi_{f,\mathfrak{p}}$ of the automorphic representation π_f is Steinberg or its twist.

Theorem 2.3. Let f be a Hilbert newform in $S_{k,w}^{n,o}(K_1(N) \cap K_{11}(p^s); \mathcal{O})$ as above. Assume f is not exceptional. Let ρ be a character $\rho : \Gamma \longrightarrow O_f^{\times}$ as above. Then the kernel and the cokernel of the map

$$X(T_f \otimes T_{\rho}/F_{\text{cyc}})_{\Gamma_n} \longrightarrow X(T_f \otimes T_{\rho}/F_n)$$

are finite groups for all n with their cardinality uniformly bounded independent of n .

Proof. Let v_n be a prime of F_n lying above S and let v_c be a prime of F_{cyc} lying above it. Given such a prime v_n we fix a prime \bar{v} in $\bar{\mathbb{Q}}$ lying above it. Recall from § 1.2, G_{v_n} (resp. G_{v_c}) denotes the decomposition subgroup $\bar{\mathbb{Q}}/F_n$ (resp. $\bar{\mathbb{Q}}/F_{\text{cyc}}$) for the prime \bar{v}/v_n (resp. \bar{v}/v_{∞}). The corresponding inertia subgroups are I_{v_n} and I_{v_c} respectively. Similarly, G_{v_c/v_n} (resp. I_{v_c/v_n}) denotes the decomposition subgroup (resp. inertia subgroup) of the Galois group of F_{cyc}/F_n with respect to the primes v_{∞}/v_n . Also the various Frobenius elements are given by $\langle \text{Fr}_{\mathfrak{p}} \rangle := \frac{G_{\mathfrak{p}}}{I_{\mathfrak{p}}}$, $\langle \text{Fr}_{v_n} \rangle := \frac{G_{v_n}}{I_{v_n}}$ and $\langle \text{Fr}_{v_c/v_n} \rangle := \frac{G_{v_c/v_n}}{I_{v_c/v_n}}$. Set

$$J_{v_n} = \begin{cases} H^1(I_{v_n}, T_{\rho} \otimes A_f) & \text{if } v_n \mid S, v_n \nmid p \\ H^1(I_{v_n}, T_{\rho} \otimes (A_f)_{\mathfrak{p}}^-)^{G_{v_n}} & \text{if } v_n \mid \mathfrak{p} \mid p, \end{cases}$$

$$J_{v_c} = \begin{cases} H^1(I_{v_c}, T_{\rho} \otimes A_f) & \text{if } v_c \in S, v_c \nmid p \\ H^1(I_{v_c}, T_{\rho} \otimes (A_f)_{\mathfrak{p}}^-)^{G_{v_c}} & \text{if } v_c \mid \mathfrak{p} \mid p, \end{cases}$$

We study the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & S(T_\rho \otimes A_f / F_{\text{cyc}})^{\Gamma_n} & \longrightarrow & H^1(F_S / F_{\text{cyc}}, T_\rho \otimes A_f)^{\Gamma_n} & \longrightarrow & (\bigoplus_{v_c | S} J_{v_c})^{\Gamma_n} \\
& & \uparrow \alpha_n & & \uparrow \phi_n & & \uparrow \theta_n = \bigoplus \theta_{v_n} \\
0 & \longrightarrow & S(T_\rho \otimes A_f / F_n) & \longrightarrow & H^1(F_S / F_n, T_\rho \otimes A_f) & \longrightarrow & \bigoplus_{v_n | S} J_{v_n}
\end{array} \quad (15)$$

First, we prove that $\ker(\alpha_n)$ is finite for all n . We will show $\ker(\phi_n)$ is finite for all n . Note that $\ker(\phi_n) \cong H^1(\Gamma_n, (T_\rho \otimes A_f)^{G_{F_{\text{cyc}}}})$. As Γ_n is topologically cyclic and $T_\rho \otimes A_f$ is a cofinitely generated \mathbb{Z}_p module it follows that $\ker(\phi_n)$ is finite if and only if $H^0(\Gamma_n, (T_\rho \otimes A_f)^{G_{F_{\text{cyc}}}})$ is finite. Also, to show $(T_\rho \otimes A_f)^{G_{F_n}}$ is finite, it suffices to show $V_f(\rho)^{G_{F_n}} = 0$. If $V_f(\rho)^{G_{F_n}} \neq 0$, then $V_f(\rho)$ contains a trivial G_{F_n} sub-representation. In that case, we choose a place $\lambda \nmid Np$ and then restrict G_F representation to Fr_λ . Then considering the eigenvalues of Fr_λ , we immediately get a contradiction by Theorem 1.5(1)(a), Theorem 1.5(1)(b).

As $H^0(F_n, V_f(\rho)) = 0$ for every n and $\ker(\alpha_n)$ is finite for all n , we deduce from [Oc2, Theorem 3.5(i)] that $\ker(\alpha_n)$ is uniformly bounded independent of n .

For $\text{coker}(\alpha_n)$, first note that $\text{coker}(\phi_n) \subset H^2(\Gamma_n, (T_\rho \otimes A_f)^{G_{F_{\text{cyc}}}}) = 0$ as p -cohomological dimension of $\Gamma_n = 1$ for any n . Using the Snake lemma, it suffices to show that the kernel of θ_n are finite and uniformly bounded independent of n . Now, there are only finite number of primes in F_{cyc} lying above a given prime in any F_n . Hence it is enough to prove $\ker(\theta_{v_n})$ is finite for each $v_n \mid S$. Now for a prime $v_n \mid S$ such that $v_n \nmid p$, we have

$$\ker(H^1(I_{v_n}, T_\rho \otimes A_f) \longrightarrow H^1(I_{v_c}, T_\rho \otimes A_f)) \cong H^1(I_{v_n}/I_{v_c}, T_\rho \otimes A_f^{I_{v_c}}). \quad (16)$$

The last isomorphism follows from the inflation-restriction sequence of Galois cohomology. We know that the cyclotomic \mathbb{Z}_p extension of any number field is unramified outside primes above p . Thus $I_{v_n}/I_{v_c} \cong I_{v_c/v_n} = 0$ whenever $v_n \nmid p$. Using (16) in diagram (15), it is immediate that $\ker(\theta_{v_n})$ vanishes for $v_n \nmid p$.

To consider $\ker(\theta_{v_n})$ for primes $v_n \mid \mathfrak{p} \mid p$, we study

$$\begin{aligned}
\ker(H^1(I_{v_n}, T_\rho \otimes (A_f)_\mathfrak{p}^-)^{G_{v_n}} &\longrightarrow H^1(I_{v_c}, T_\rho \otimes (A_f)_\mathfrak{p}^-)^{G_{v_c}}) \\
&\cong H^1(I_{v_c/v_n}, (T_\rho \otimes (A_f)_\mathfrak{p}^-)^{G_{v_c}})^{<\text{Fr}_{v_c/v_n}>}
\end{aligned} \quad (17)$$

As, ρ acts trivially on G_{v_c} ,

$$H^1(I_{v_c/v_n}, (T_\rho \otimes (A_f)_\mathfrak{p}^-)^{G_{v_c}}) \cong H^1(I_{v_c/v_n}, T_\rho \otimes ((A_f)_\mathfrak{p}^-)^{G_{v_c}}).$$

Now F_{cyc}/F is totally ramified at \mathfrak{p} for all $\mathfrak{p} \mid p$. Hence, $I_{v_c/v_n} \cong \mathbb{Z}_p$ for every v_n lying above $\mathfrak{p} \mid p$. Note that, as an abstract group $T_\rho \cong O_f$, $((A_f)_\mathfrak{p}^-)^\vee \cong O_f$. Hence

$$H^1(I_{v_c/v_n}, T_\rho \otimes (A_f)_\mathfrak{p}^-)^{G_{v_c}} = 0 \text{ unless } I_{v_c/v_n} \text{ acts trivially on } T_\rho \otimes ((A_f)_\mathfrak{p}^-)^{G_{v_c}}.$$

Thus it suffices to consider the case where I_{v_c/v_n} is acting trivially on $T_\rho \otimes ((A_f)_\mathfrak{p}^-)^{G_{v_c}}$, as otherwise kernel in (17) is 0. Then

$$H^1(I_{v_c/v_n}, T_\rho \otimes (A_f)_\mathfrak{p}^-)^{G_{v_c}} \cong \text{Hom}(I_{v_c/v_n}, (T_\rho \otimes (A_f)_\mathfrak{p}^-)^{G_{v_c}}).$$

As F_{cyc}/F is abelian, the action of the Frobenius $\langle \text{Fr}_{v_c/v_n} \rangle$ on I_{v_c/v_n} (via lifting and conjugation) is trivial. Hence, the module in (17) is isomorphic to

$$\begin{aligned} & \text{Hom}_{\langle \text{Fr}_{v_c/v_n} \rangle}(I_{v_c/v_n}, (T_\rho \otimes (A_f)_\mathfrak{p}^-)^{G_{v_c}}) \\ & \cong \text{Hom}(I_{v_c/v_n}, (T_\rho \otimes (A_f)_\mathfrak{p}^-)^{G_{v_n}}) \\ & \cong \text{Hom}(I_{v_c/v_n}, (T_\rho \otimes (A_f)_\mathfrak{p}^-)^{\frac{G_{v_n}}{I_{v_n}}}) \end{aligned}$$

We claim that $(T_\rho \otimes (A_f)_\mathfrak{p}^-)^{\langle \text{Fr}_{v_n} \rangle}$ is finite. On the one dimensional vector space corresponding to $(A_f)_\mathfrak{p}^-$, $\text{Fr}_\mathfrak{p}$ acts by multiplication by $\alpha_\mathfrak{p}(f)$. By our assumption in this theorem that f is not exceptional and remark 2.2, it follows that the eigenvalue of $\text{Fr}_\mathfrak{p}$ acting on the 1 dimensional line corresponding to $(A_f)_\mathfrak{p}^-$ is not a root of unity. On the other hand, $\rho(g) = 1$ for any $g \in G_{F_{\text{cyc}}}$ and F_{cyc}/F is totally ramified at any prime in F lying above p . Hence the eigenvalue corresponding to the action of $\text{Fr}_\mathfrak{p}$ on T_ρ is a root of unity. Combining these facts, we deduce that the eigenvalue of $\text{Fr}_\mathfrak{p}$ on the line corresponding to $T_\rho \otimes (A_f)_\mathfrak{p}^-$ is not a root of unity. Hence Fr_{v_n} acts non-trivially on $T_\rho \otimes (A_f)_\mathfrak{p}^-$ for any n and any ρ as before. Hence $(T_\rho \otimes (A_f)_\mathfrak{p}^-)^{\langle \text{Fr}_{v_n} \rangle}$ is indeed finite. Also as $\langle \text{Fr}_{v_n} \rangle \cong \langle \text{Fr}_{v_c} \rangle$ for $n \gg 0$, we deduce that the size of $(T_\rho \otimes (A_f)_\mathfrak{p}^-)^{\langle \text{Fr}_{v_n} \rangle}$ is independent of n for large enough n . As $I_{v_c/v_n} \cong \mathbb{Z}_p$ for every n ; for any ρ and every $n \geq 0$, the module in (17), given by $\text{Hom}(I_{v_c/v_n}, (T_\rho \otimes (A_f)_\mathfrak{p}^-)^{\langle \text{Fr}_{v_n} \rangle})$, is also finite with its cardinality bounded independent of n . Hence same is true for $\ker(\theta_{v_n})$ for $v_n \mid \mathfrak{p}$. This completes the proof. \square

Let O be the ring of integers of a finite extension of \mathbb{Q}_p . Take M to be a finitely generated Λ module, where $\Lambda = O[[\Gamma]]$. Let us denote $\text{Ext}_\Lambda^1(M, \Lambda)$ by $a_\Lambda^1(M)$.

Lemma 2.4. *Let M be a finitely generated torsion $\Lambda = O[[\Gamma]]$ module such that M_{Γ_n} is finite for each n . Then*

$$a_\Lambda^1(M) \cong \varprojlim_n (M^{\vee})^{\Gamma_n}.$$

Proof. See [Pe, §1.3, page 733]. \square

Lemma 2.5. *Let M be a finitely generated torsion $\Lambda = O[[\Gamma]]$ module. Then $\text{Ch}_\Lambda(M) = \text{Ch}_\Lambda(a_\Lambda^1(M))$, considered as ideals in $O[[\Gamma]]$.*

Proof. See [J-P, Lemma 3.5]. \square

Lemma 2.6. *Let M be a finitely generated torsion $\Lambda = O[[\Gamma]]$ module. Then there exists a character $\rho : \Gamma = \text{Gal}(F_{\text{cyc}}/F) \rightarrow \text{Aut}(O)$ such that $(M(\rho))_{\Gamma_n}$ is finite for every n , where $M(\rho)$ is as defined in section 1.2.*

Proof. This is well known, for example see [Pe, §2.6, Page 740]. \square

Lemma 2.7. *For any cuspidal Hilbert newform f , the dual Selmer groups $X(T_f/F_{\text{cyc}})$ and $X(T_f^*/F_{\text{cyc}})$ are finitely generated $O_f[[\Gamma]]$ modules.*

Proof. The proof is similar to the one in elliptic modular form case (see for example [J-P, Lemma 3.7]).

Throughout the rest of the article we make this assumption -

Hypothesis 2.8. $(\mathbf{Tor}) = (\mathbf{Tor}_f)$: For any cuspidal Hilbert newform f , $X(T_f/F_{\text{cyc}})$ is finitely generated torsion $O_f[[\Gamma]]$ module.

Corollary 2.9. It follows from remark 1.20 that by hypothesis (\mathbf{Tor}) , we have for any cuspidal Hilbert newform f , $X(T_f^*/F_{\text{cyc}})$ is also torsion over $O_f[[\Gamma]]$.

Theorem 2.10. Let the notation be as before. Let $f \in S_{k,w}^{n,o}(K_1(N) \cap K_{11}(p^s); \mathcal{O})$ be a Hilbert newform nearly ordinary at $\mathfrak{p}|p$, which is not exceptional (as defined in theorem 2.3). Assume (\mathbf{Tor}) holds. Then the functional equation holds for $X(T_f/F_{\text{cyc}})$ i.e. we have the equality of ideals in $O_f[[\Gamma]]$,

$$Ch_{O_f[[\Gamma]]}(X(T_f/F_{\text{cyc}})) = Ch_{O_f[[\Gamma]]}(X(T_f^*/F_{\text{cyc}})^\iota).$$

Proof. By the assumption (\mathbf{Tor}) and corollary 2.9, both $X(T_f/F_{\text{cyc}})$ and $X(T_f^*/F_{\text{cyc}})$ are torsion over $O_f[[\Gamma]]$. Thus we can find a ρ by lemma 2.6 such that $X(T_f \otimes T_\rho/F_n)$ and $X(T_f^* \otimes T_{\rho^{-1}}/F_n)$ are both finite groups for every n . Then from the generalized Cassels-Tate pairing of Flach (see [Pe, 3.1.1]), we obtain that $S(T_{\rho^{-1}} \otimes A_f^*/F_n) \cong X(T_\rho \otimes T_f/F_n)$ for every n . Hence we get

$$X(T_\rho \otimes T_f/F_{\text{cyc}}) \cong \varprojlim_n X(T_\rho \otimes T_f/F_n) \cong \varprojlim_n S(T_{\rho^{-1}} \otimes A_f^*/F_n). \quad (18)$$

By Theorem 2.3 and remark 1.20, we see that the kernel and the cokernel of the natural restriction map, given by $S(T_{\rho^{-1}} \otimes A_f^*/F_n) \xrightarrow{\alpha_n^*} S(T_{\rho^{-1}} \otimes A_f^*/F_{\text{cyc}})^{\Gamma_n}$, are finite groups and their size uniformly bounded independent of n . Thus we obtain from (18) that the induced map

$$X(T_\rho \otimes T_f/F_{\text{cyc}}) \xrightarrow{\phi_\rho} \varprojlim_n S(T_{\rho^{-1}} \otimes A_f^*/F_{\text{cyc}})^{\Gamma_n} \quad (19)$$

is a $O_f[[\Gamma]]$ pseudo-isomorphism. We have

$$\varprojlim_n S(T_{\rho^{-1}} \otimes A_f^*/F_{\text{cyc}})^{\Gamma_n} = \varprojlim_n (X(T_{\rho^{-1}} \otimes T_f^*/F_{\text{cyc}})^\vee)^{\Gamma_n} \stackrel{\text{Lemma 2.4}}{\cong} a_\Lambda^1(X(T_{\rho^{-1}} \otimes T_f^*/F_{\text{cyc}})^\iota).$$

Combining this with (19) we get an $O_f[[\Gamma]]$ module pseudo-isomorphism

$$X(T_f \otimes T_\rho/F_{\text{cyc}}) \xrightarrow{\theta_\rho} a_\Lambda^1(X(T_f^* \otimes T_{\rho^{-1}}/F_{\text{cyc}})^\iota). \quad (20)$$

We recall from (14), $X(T_f \otimes T_\rho/F_{\text{cyc}}) \cong X(T_f/F_{\text{cyc}}) \otimes T_{\rho^{-1}}$. On the other hand, we have $a_\Lambda^1(X(T_f^* \otimes T_{\rho^{-1}}/F_{\text{cyc}})^\iota) \cong a_\Lambda^1(X(T_f^*/F_{\text{cyc}})^\iota) \otimes T_{\rho^{-1}}$ (see [Pe, Page 744, §3.2.1-3.2.2]). Thus tensoring (20) with T_ρ , we get a pseudo-isomorphism of $O_f[[\Gamma]]$ modules

$$X(T_f/F_{\text{cyc}}) \longrightarrow a_\Lambda^1(X(T_f^*/F_{\text{cyc}})^\iota)$$

which is independent of ρ . Hence $Ch_{O_f[[\Gamma]]}(X(T_f/F_{\text{cyc}})) = Ch_{O_f[[\Gamma]]}(a_\Lambda^1(X(T_f^*/F_{\text{cyc}})^\iota))$ as ideals in $O_f[[\Gamma]]$. By applying Lemma 2.5, we get that

$$Ch_{O_f[[\Gamma]]}(X(T_f/F_{\text{cyc}})) = Ch_{O_f[[\Gamma]]}(X(T_f^*/F_{\text{cyc}})^\iota). \quad \square$$

3. FUNCTIONAL EQUATION FOR A NEARLY ORDINARY HIDA FAMILY

We begin by proving a specialization result relating the ‘big’ Selmer group with the individual Selmer groups.

Theorem 3.1. *Assume **(Irr)**, **(Dist)**. Let \mathcal{R} be a branch of $\mathbf{H}_{\mathcal{N}, \mathcal{O}}$ and assume that \mathcal{R} is a power series ring in many variable (i.e. we assume that $\mathcal{R} \cong O[[X_1, \dots, X_r]]$, where $r = d + 1 + \delta_{F,p}$, here $\delta_{F,p}$ is the defect of Leopoldt’s conjecture for F at p , $d = [F : \mathbb{Q}]$, and O is the ring of integer of some finite extension of \mathbb{Q}_p). Let s_ξ^\vee be the natural specialization map*

$$\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})/P_\xi \mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}}) \xrightarrow{s_\xi^\vee} X(T_{f_\xi}/F_{\text{cyc}}) \quad (21)$$

- Then the kernel and the cokernel of s_ξ^\vee are finitely generated \mathbb{Z}_p modules for every $\xi \in \mathfrak{X}(\mathcal{R})$ and
- There exists a non zero ideal J in \mathcal{R} such that for any $\xi \in \mathfrak{X}(\mathcal{R}) \setminus S_J$, the kernel and the cokernel of s_ξ^\vee are finite, where the set S_J is defined as $S_J := \{\xi \in \mathfrak{X}(\mathcal{R}) \mid P_\xi \text{ does not contain } J\}$.

In particular, assuming **(Tor)**, the equality

$$\text{Ch}_{O_{f_\xi}[[T]]}(\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})/P_\xi \mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})) = \text{Ch}_{O_{f_\xi}[[T]]}(X(T_{f_\xi}/F_{\text{cyc}}))$$

holds for all $\xi \in \mathfrak{X}(\mathcal{R}) \setminus S_J$.

Proof. For a finitely generated \mathcal{R} module M , we define $M^\dagger := \text{Hom}_{\mathcal{R}}(M, \mathcal{R})$. For convenience of notation, let $\mathcal{T} = \mathcal{T}_{\mathcal{R}}$ and $\mathcal{A} = \mathcal{A}_{\mathcal{R}}$. As before, v_c will denote a prime in F_{cyc} lying above S and I_c denotes the inertia subgroup of $\mathbb{Q}/F_{\text{cyc}}$ with respect to the primes \bar{v}/v_c . We have the commutative diagram with the natural maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & S(\mathcal{A}/F_{\text{cyc}})[P_\xi] & \longrightarrow & H^1(F_S/F_{\text{cyc}}, \mathcal{A})[P_\xi] & \longrightarrow & \bigoplus_{v_c | S, v_c \nmid p} H^1(I_{v_c}, \mathcal{A})[P_\xi] \oplus \bigoplus_{v_c | \mathfrak{p} | p} H^1(I_{v_c}, \mathcal{A}_{\mathfrak{p}}^-)[P_\xi] \\ & & \uparrow s_\xi & & \uparrow \eta_\xi & & \uparrow \delta_\xi = \oplus \delta_{v_c}^\xi \\ 0 & \longrightarrow & S(\mathcal{A}_{f_\xi}/F_{\text{cyc}}) & \longrightarrow & H^1(F_S/F_{\text{cyc}}, \mathcal{A}_{f_\xi}) & \longrightarrow & \bigoplus_{v_c | S, v_c \nmid p} H^1(I_{v_c}, \mathcal{A}_{f_\xi}) \oplus \bigoplus_{v_c | \mathfrak{p} | p} H^1(I_{v_c}, (\mathcal{A}_{f_\xi})_{\mathfrak{p}}^-) \end{array} \quad (22)$$

Recall, $\mathcal{A}_{f_\xi} \cong \mathcal{A}[P_\xi]$ and $(\mathcal{A}_{f_\xi})_{\mathfrak{p}}^- \cong \mathcal{A}_{\mathfrak{p}}^-[P_\xi]$. Since we assume that the residual representation is absolutely irreducible, we have (by [Gr2, Remark 3.4.1])

$$H^1(F_S/F_{\text{cyc}}, \mathcal{A}[P_\xi]) \xrightarrow{\eta_\xi} H^1(F_S/F_{\text{cyc}}, \mathcal{A})[P_\xi] \quad (23)$$

is an isomorphism. By Snake lemma, we get that $\ker(s_\xi)$ is trivial for every ξ . Thus $\text{coker}(s_\xi^\vee) = 0$.

Next we want to prove $\ker(s_\xi^\vee)$ is finitely generated \mathbb{Z}_p module for every ξ . As there are only finitely many primes in F_{cyc} lying over a given prime in F , it is enough to show that $\ker(\delta_{v_c}^\xi)^\vee$ is finitely generated \mathbb{Z}_p module for all $v_c \mid S$. By our assumption, $\mathcal{R} \cong O[[X_1, \dots, X_r]]$, where $r = [F : \mathbb{Q}] + 1 + \delta_{F,p}$, where $\delta_{F,p}$ is the defect of Leopoldt’s conjecture for F at p and O is some finite extension of \mathbb{Z}_p . Since \mathcal{R} is regular local and P_ξ is a prime ideal of height r , we have $P_\xi = (x_1, \dots, x_r)$, here x_1, \dots, x_r is a regular sequence of prime elements of \mathcal{R} . Define $P_0 = (0)$ and

$P_i = (x_1, \dots, x_i)$, for $1 \leq i \leq r$. Then P_i is a prime ideal and $\mathcal{A}[P_i]$ is divisible as \mathcal{R}/P_i module. Notice that $\mathcal{A}[P_i] = (\mathcal{A}[P_{i-1}])[x_i]$, and multiplication by x_i is surjective on $\mathcal{A}[P_{i-1}]$. We get a induced map of \mathcal{R}/P_i modules

$$0 \rightarrow \mathcal{A}^{I_{v_c}}[P_{i-1}]/x_i \mathcal{A}^{I_{v_c}}[P_{i-1}] \rightarrow H^1(I_{v_c}, \mathcal{A}[P_i]) \rightarrow H^1(I_{v_\infty}, \mathcal{A}[P_{i-1}])[x_i] \rightarrow 0.$$

Thus we can obtain kernel of $\delta_{v_c}^\xi$ via successive extensions. Let $\text{Ker}_1 = \mathcal{A}^{I_{v_c}}/x_1 \mathcal{A}^{I_{v_c}}$, and let Ker_i denotes the kernel of the map $H^1(I_{v_c}, \mathcal{A}[P_i]) \rightarrow H^1(I_{v_c}, \mathcal{A})[P_i]$, then Ker_i is an extension of the form,

$$0 \rightarrow \mathcal{A}^{I_{v_c}}[P_{i-1}]/x_i \mathcal{A}^{I_{v_c}}[P_{i-1}] \rightarrow \text{Ker}_i \rightarrow \text{Ker}_{(i-1)}[x_i] \rightarrow 0. \quad (24)$$

Then in this notation $\ker(\delta_{v_c}^\xi) = \text{Ker}_r$.

First, we show that $(\text{Ker}_r)^\vee$ is finitely generated \mathbb{Z}_p module for every ξ . We also denote I_{v_c} by G for the rest of this proof. Taking Pontryagin dual, we get from (24), for each i ,

$$0 \rightarrow \frac{\text{Ker}_{i-1}^\vee}{x_i \text{Ker}_{i-1}^\vee} \rightarrow \text{Ker}_i^\vee \rightarrow \frac{\mathcal{T}_G^\dagger}{P_{i-1} \mathcal{T}_G^\dagger}[x_i] \rightarrow 0. \quad (25)$$

Now for $i = 1$, $\text{Ker}_1^\vee \cong \mathcal{T}_G^\dagger[x_1]$ is a finitely generated \mathcal{R}/x_1 module. By induction, assume that for $i = 1, 2, \dots, r-1$, Ker_i^\vee is a finitely generated $\mathcal{R}/(x_1, \dots, x_i)$ module. Also, \mathcal{T}_G^\dagger being a finitely generated \mathcal{R} module, it is immediate that $\mathcal{T}_G^\dagger/P_{r-1} \mathcal{T}_G^\dagger$ is a finitely generated \mathcal{R}/P_{r-1} module. Consequently, $\mathcal{T}_G^\dagger/P_{r-1} \mathcal{T}_G^\dagger[x_r]$ is a finitely generated \mathcal{R}/P_r module. Also by induction hypothesis, $\text{Ker}_{r-1}^\vee/x_r$ is a finitely generated $\mathcal{R}/(P_{r-1}, x_r) \cong \mathcal{R}/P_r$ module. Hence, we deduce from (25), that Ker_r^\vee is a finitely generated \mathcal{R}/P_r module. Recall, in this notation, $P_r = (x_1, \dots, x_r) = P_\xi$ and hence $\mathcal{R}/P_\xi \cong O$, a finite extension of \mathbb{Z}_p . This finishes the proof of the first assertion of the theorem.

For the second assertion of the theorem, we have to show that there exists a non zero ideal J in \mathcal{R} such that for any $\xi \in \mathfrak{X}(\mathcal{R})$ such that $\ker(\xi) = P_\xi \not\supseteq J$, the rank

$$\text{rk}_{\mathcal{R}/(x_1, \dots, x_r)} \text{Ker}_r^\vee = 0. \quad (26)$$

By [Gr2, Theorem 2.1], corresponding to the \mathcal{R} module \mathcal{T}_G^\dagger there exists an ideal $J \neq 0$ such that the following two equations hold

$$\text{rk}_{\mathcal{R}/P} \mathcal{T}_G^\dagger[P] = 0 \text{ for any prime ideal } P \not\supseteq J \quad (27)$$

$$\text{rk}_{\mathcal{R}/P} \frac{\mathcal{T}_G^\dagger}{P \mathcal{T}_G^\dagger} = \text{rk}_{\mathcal{R}} \mathcal{T}_G^\dagger \text{ for any prime ideal } P \not\supseteq J \quad (28)$$

We will go on to show that this J will work for us. Let $P_\xi = (x_1, \dots, x_r) \not\supseteq J$. Then as $(x_1) \not\supseteq J$, we have

$$\text{rk}_{\mathcal{R}/x_1} \mathcal{T}_G^\dagger[x_1] = 0.$$

But $\text{Ker}_1^\vee \cong \mathcal{T}_G^\dagger[x_1]$. Thus we get that

$$\text{rk}_{\mathcal{R}/x_1} \text{Ker}_1^\vee = 0.$$

Let us assume for $i = 1, 2, \dots, r-1$,

$$\text{rk}_{\mathcal{R}/(x_1, \dots, x_i)} \text{Ker}_i^\vee = 0. \quad (29)$$

Notice that

$$\begin{aligned} \text{rk}_{\mathcal{R}/(x_1, \dots, x_r)} \frac{\mathcal{T}_G^\dagger}{P_{r-1}T_G^\dagger}[x_r] &= \text{rk}_{\mathcal{R}/(x_1, \dots, x_r)} \frac{\mathcal{T}_G^\dagger}{(P_{r-1}, x_r)T_G^\dagger} - \text{rk}_{\mathcal{R}/(x_1, \dots, x_{r-1})} \frac{\mathcal{T}_G^\dagger}{P_{r-1}T_G^\dagger} \\ &= \text{rk}_{\mathcal{R}/(x_1, \dots, x_r)} \frac{\mathcal{T}_G^\dagger}{P_r T_G^\dagger} - \text{rk}_{\mathcal{R}/(x_1, \dots, x_{r-1})} \frac{\mathcal{T}_G^\dagger}{P_{r-1}T_G^\dagger}. \end{aligned} \quad (30)$$

By our assumption that $P_r \not\subseteq J$, we deduce from (27) that

$$\text{rk}_{\mathcal{R}/P_r} \frac{\mathcal{T}_G^\dagger}{P_r T_G^\dagger} = \text{rk}_{\mathcal{R}/P_{r-1}} \frac{\mathcal{T}_G^\dagger}{P_{r-1}T_G^\dagger} = \text{rk}_{\mathcal{R}} \mathcal{T}_G^\dagger.$$

Hence we deduce from (30) that $\text{rk}_{\mathcal{R}/P_r} \frac{\mathcal{T}_G^\dagger}{P_{r-1}T_G^\dagger}[x_r] = 0$.

Thus using (25), to finish the proof of the theorem, it suffices to show that

$$\text{rk}_{\mathcal{R}/P_r} \text{Ker}_{r-1}^\vee / x_r = 0. \quad (31)$$

Now

$$\begin{aligned} \text{rk}_{\mathcal{R}/(x_1, \dots, x_r)} \frac{\text{Ker}_{r-1}^\vee}{x_r \text{Ker}_{r-1}^\vee} &= \text{rk}_{\mathcal{R}/(x_1, \dots, x_{r-1})} \text{Ker}_{r-1}^\vee + \text{rk}_{\mathcal{R}/(x_1, \dots, x_r)} \text{Ker}_{r-1}^\vee[x_r] \\ &= \text{rk}_{\mathcal{R}/(x_1, \dots, x_r)} \text{Ker}_{r-1}^\vee[x_r], \end{aligned} \quad (32)$$

where the last equality follows from (29). Thus to prove (31), we are further reduced to showing

$$\text{rk}_{\mathcal{R}/(x_1, \dots, x_r)} \text{Ker}_{r-1}^\vee[x_r] = 0 \quad (33)$$

Now we have the exact sequence,

$$0 \rightarrow \frac{\text{Ker}_{r-2}^\vee}{x_{r-1} \text{Ker}_{r-2}^\vee}[x_r] \rightarrow \text{Ker}_{r-1}^\vee[x_r] \rightarrow \frac{\mathcal{T}_G^\dagger}{P_{r-2}T_G^\dagger}[(x_{r-1}, x_r)] \quad (34)$$

Proceeding similarly as in (30), we deduce that

$$\begin{aligned} \text{rk}_{\mathcal{R}/P_r} \frac{\mathcal{T}_G^\dagger}{P_{r-2}T_G^\dagger}[(x_{r-1}, x_r)] &= \text{rk}_{\mathcal{R}/P_r} \frac{\mathcal{T}_G^\dagger}{P_r T_G^\dagger} - \text{rk}_{\mathcal{R}/P_{r-2}} \frac{\mathcal{T}_G^\dagger}{P_{r-2}T_G^\dagger} \\ &= 0. \end{aligned} \quad (35)$$

Thus

$$\begin{aligned} \text{rk}_{\mathcal{R}/P_r} \text{Ker}_{r-1}^\vee[x_r] &= \text{rk}_{\mathcal{R}/P_r} \frac{\text{Ker}_{r-2}^\vee}{x_{r-1} \text{Ker}_{r-2}^\vee}[x_r] \\ &= \text{rk}_{\mathcal{R}/P_r} \frac{\text{Ker}_{r-2}^\vee}{(x_{r-1}, x_r) \text{Ker}_{r-2}^\vee} - \text{rk}_{\mathcal{R}/P_{r-1}} \frac{\text{Ker}_{r-2}^\vee}{x_{r-1} \text{Ker}_{r-2}^\vee}. \end{aligned} \quad (36)$$

Now, note that

$$\begin{aligned} \text{rk}_{\mathcal{R}/P_r} \frac{\text{Ker}_{r-2}^\vee}{(x_{r-1}, x_r) \text{Ker}_{r-2}^\vee} &= \text{rk}_{\mathcal{R}/P_{r-2}} \text{Ker}_{r-2}^\vee + \text{rk}_{\mathcal{R}/P_r} \text{Ker}_{r-2}^\vee[(x_{r-1}, x_r)] \\ &= \text{rk}_{\mathcal{R}/P_r} \text{Ker}_{r-2}^\vee[(x_{r-1}, x_r)]. \end{aligned} \quad (37)$$

Here the last equality follows from the induction hypothesis in (29). Using this in (36), we deduce that

$$\mathrm{rk}_{\mathcal{R}/P_r} \mathrm{Ker}_{r-1}^\vee[x_r] = \mathrm{rk}_{\mathcal{R}/P_r} \mathrm{Ker}_{r-2}^\vee[(x_{r-1}, x_r)] - \mathrm{rk}_{\mathcal{R}/P_{r-1}} \frac{\mathrm{Ker}_{r-2}^\vee}{x_{r-1} \mathrm{Ker}_{r-2}^\vee} \quad (38)$$

Recall, from (25), for $i = r - 1$, we have the exact sequence

$$0 \rightarrow \frac{\mathrm{Ker}_{r-2}^\vee}{x_{r-1} \mathrm{Ker}_{r-2}^\vee} \rightarrow \mathrm{Ker}_{r-1}^\vee \rightarrow \frac{\mathcal{T}_G^\dagger}{P_{r-2} \mathcal{T}_G^\dagger}[x_{r-1}] \rightarrow 0. \quad (39)$$

Using induction hypothesis (29) in (39), we deduce that

$$\mathrm{rk}_{\mathcal{R}/P_{r-1}} \frac{\mathrm{Ker}_{r-2}^\vee}{x_{r-1} \mathrm{Ker}_{r-2}^\vee} = 0.$$

Hence we obtain from (38) that

$$\mathrm{rk}_{\mathcal{R}/P_r} \mathrm{Ker}_{r-1}^\vee[x_r] = \mathrm{rk}_{\mathcal{R}/P_r} \mathrm{Ker}_{r-2}^\vee[(x_{r-1}, x_r)] \quad (40)$$

Proceeding in a similar way, we deduce that

$$\begin{aligned} \mathrm{rk}_{\mathcal{R}/P_r} \mathrm{Ker}_r^\vee &= \mathrm{rk}_{\mathcal{R}/P_r} \mathrm{Ker}_{r-1}^\vee[x_r] \\ &= \mathrm{rk}_{\mathcal{R}/P_r} \mathrm{Ker}_{r-2}^\vee[(x_{r-1}, x_r)] \\ &= \mathrm{rk}_{\mathcal{R}/P_r} \mathrm{Ker}_1^\vee[(x_2, \dots, x_r)] \\ &= \mathrm{rk}_{\mathcal{R}/P_r} \mathcal{T}_G^\dagger[x_1][(x_2, \dots, x_r)] \\ &= \mathrm{rk}_{\mathcal{R}/P_r} \mathcal{T}_G^\dagger[(x_1, x_2, \dots, x_r)] \\ &= \mathrm{rk}_{\mathcal{R}/P_r} \mathcal{T}_G^\dagger[(x_1, x_2, \dots, x_r)] \\ &= \mathrm{rk}_{\mathcal{R}/P_r} \mathcal{T}_G^\dagger[P_r] = 0. \end{aligned} \quad (41)$$

Here the last equality follows from our hypothesis that $P_r = P_\xi \not\supseteq J$. This finishes the proof of the second assertion of the theorem. \square

Remark 3.2. Let the assumptions be as in Theorem 3.1. Then proceeding as in the proof of Theorem 3.1, we can deduce the corresponding theorem for $\mathcal{T}_\mathcal{R}^*$ i.e. there exists a non zero ideal J^* in \mathcal{R} such that for any $\xi \in \mathfrak{X}(\mathcal{R}) \setminus S_{J^*}$, the kernel and the cokernel of $s_\xi^{*\vee}$ are finite, where $S_{J^*} := \{\xi \in \mathfrak{X}(\mathcal{R}) \mid P_\xi \text{ does not contain } J^*\}$. Consequently under the assumption **(Tor)**,

$$\mathrm{Ch}_{O_{f_\xi}[[\Gamma]]}(\mathcal{X}(\mathcal{T}_\mathcal{R}^*/F_{\mathrm{cyc}})/P_\xi \mathcal{X}(\mathcal{T}_\mathcal{R}^*/F_{\mathrm{cyc}})) = \mathrm{Ch}_{O_{f_\xi}[[\Gamma]]}(X(T_{f_\xi}^*/F_{\mathrm{cyc}}))$$

for every $\xi \in \mathfrak{X}(\mathcal{R}) \setminus S_{J^*}$.

Remark 3.3. Under the assumption **(Tor)**, by applying the involution ι we obtain,

$$\begin{aligned} \mathrm{Ch}_{O_{f_\xi}[[\Gamma]]}(\mathcal{X}(\mathcal{T}_\mathcal{R}^*/F_{\mathrm{cyc}})^\iota/P_\xi \mathcal{X}(\mathcal{T}_\mathcal{R}^*/F_{\mathrm{cyc}})^\iota) &= \mathrm{Ch}_{O_{f_\xi}[[\Gamma]]}((\mathcal{X}(\mathcal{T}_\mathcal{R}^*/F_{\mathrm{cyc}})/P_\xi \mathcal{X}(\mathcal{T}_\mathcal{R}^*/F_{\mathrm{cyc}}))^\iota) \\ &= \mathrm{Ch}_{O_{f_\xi}[[\Gamma]]}(X(T_{f_\xi}^*/F_{\mathrm{cyc}})^\iota). \end{aligned}$$

for every $\xi \in \mathfrak{X}(\mathcal{R}) \setminus S_{J^*}$.

Lemma 3.4. Assume **(Irr)**, **(Dist)** and that \mathcal{R} is a power series ring in many variables. Then $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})$ and $\mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/F_{\text{cyc}})$ are finitely generated $\mathcal{R}[[\Gamma]]$ modules.

Proof. This result follows from Theorem 3.1, remark 3.2, Lemma 2.7 and topological Nakayama's Lemma [N-S-W, Corollary 5.2.18]. \square

Corollary 3.5. Let the assumption be as in Theorem 3.1. Then under hypothesis **(Tor)**, $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})$ and $\mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/F_{\text{cyc}})$ are finitely generated torsion $\mathcal{R}[[\Gamma]]$ modules.

Proof. We will prove for $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})$ and a similar argument works for $\mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/F_{\text{cyc}})$. Choose any $\xi \in \mathfrak{X}(\mathcal{R}) \setminus S_J$. It suffices to show the localization at $\mathcal{R} \setminus P_{\xi}$, $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})_{(P_{\xi})} = 0$. By **(Tor)** and Theorem 3.1, $\frac{\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})}{P_{\xi}\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})}$ is a torsion \mathcal{R}/P_{ξ} module. From this, using localization argument and Nakayama's Lemma, we get $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})_{(P_{\xi})} = 0$. \square

Proposition 3.6. Let M and N be two finitely generated torsion modules over the $r + 1$ variable power series ring $R_O = O[[W, T]]$ where $W = (X_1, \dots, X_r)$, O is the ring of integer of a finite extension of \mathbb{Q}_p . Let $\{l_i\}_{i \in \mathbb{N}}$ be an infinite set of co-height 1 prime ideals in $O[[W]]$ such that

- (1) $O[[W]]/l_i$ is a finite extension of O , for any i .
- (2) For each l_i , both $M/l_i M$ and $N/l_i N$ are torsion over R_O/l_i and
- (3) for every i , the image of $\text{Ch}_{R_O}(M)$ (resp. $\text{Ch}_{R_O}(N)$) in R_O/l_i equals $\text{Ch}_{R_O/l_i}(M/l_i M)$ (resp. $\text{Ch}_{R_O/l_i}(N/l_i N)$), as ideals in R_O/l_i .

Then the equality of the ideals in $\text{Ch}_{R_O/l_i}(M/l_i M) = \text{Ch}_{R_O/l_i}(N/l_i N)$ in R_O/l_i for every l_i implies the equality of ideals $\text{Ch}_{R_O}(M) = \text{Ch}_{R_O}(N)$ in R_O .

Proof. For $r = 1$, the result is essentially continued in [Oc3, §3]. Suppose $l_i = (l_{i,1}, l_{i,2}, \dots, l_{i,r})$, denote by $l_i^{(j)} = (l_{i,1}, l_{i,2}, \dots, l_{i,j})$. Note that, $M \otimes R_O/l_i^{(j)} = (M \otimes R_O/l_i^{(j-1)}) \otimes R_O/l_{i,j}$. We claim that, if $\text{Ch}_{R_O/(l_{i,1}, l_{i,2}, \dots, l_{i,j})} M \otimes R_O/(l_{i,1}, l_{i,2}, \dots, l_{i,j}) = \text{Ch}_{R_O/(l_{i,1}, l_{i,2}, \dots, l_{i,j})} N \otimes R_O/(l_{i,1}, l_{i,2}, \dots, l_{i,j})$ is true for infinitely many $l_i^{(j)}$, for which first $(j-1)$ generators are same (that is $l_i^{(j-1)}$'s are same for all i), then we have,

$$\text{Ch}_{R_O/(l_i^{(j-1)})} M \otimes R_O/(l_i^{(j-1)}) = \text{Ch}_{R_O/(l_i^{(j-1)})} N \otimes R_O/(l_i^{(j-1)}).$$

Hence we prove the result by applying this to $j = r, r-1, \dots, 1$.

We use multivariable notation $h(\mathbb{T})$ to denote polynomial $h(X_1, \dots, X_r, T)$. As M and N are finitely generated torsion module over $(r+1)$ variable Iwasawa algebra, using the structure theorem of Iwasawa modules, we fix R_O module pseudo-isomorphisms ϕ and ψ respectively,

$$M \xrightarrow{\phi} \bigoplus_i R_O/\pi_O^{\mu_i} \oplus_j R_O/h_j(\mathbb{T})^{\lambda_j} \quad \text{and}$$

$$N \xrightarrow{\psi} \bigoplus_{i'} R_O/\pi_O^{\mu'_{i'}} \oplus_{j'} R_O/g_{j'}(\mathbb{T})^{\lambda'_{j'}}.$$

Here π_O is a uniformizing parameter for O . Set $h(\mathbb{T}) = \prod h_j(\mathbb{T})^{\lambda_j}$, $g(\mathbb{T}) = \prod g_{j'}(\mathbb{T})^{\lambda'_{j'}}$ and $\mu = \sum \mu_i$, $\mu' = \sum \mu'_{i'}$.

We will show that $Ch_{R_O/l_i^{(j-1)}}(N \otimes R_O/l_i^{(j-1)}) \subset Ch_{R_O/l_i^{(j-1)}}(M \otimes R_O/l_i^{(j-1)})$. Interchanging M and N , we will get the equality. Clearly, it suffices to show that the image of $\pi_O^{\mu'}$ is zero in $(R_O/l_i^{(j-1)})/\pi_O^\mu$ and the image of $g(\mathbb{T})$ is zero in $(R_O/l_i^{(j-1)})/h(\mathbb{T})$.

If $h(\mathbb{T})$ is a unit in $R_O/l_i^{(j-1)}$ then obviously the image of $g(\mathbb{T})$ in $(R_O/l_i^{(j-1)})/h(\mathbb{T})$ is zero. So we assume that $h(\mathbb{T})$ is not a unit in $R_O/l_i^{(j-1)}$. Then by [Oc3, Lemma 3.8] there is a finite extension O'' of O such that by a change of coordinate by a linear transform, we may assume that $h(\mathbb{T}) = u(\mathbb{T})f(T)$ where $u(\mathbb{T})$ is a unit in $R_{O''}$ and $f \in O''[[W]][T]$. Now, if necessary, we move to an extension of O'' containing both O' and O'' and denote again it by O' (abusing the notation, just to ease the burden of notation) such that $R_{O'}/l_i \cong O'[[T]]$. Then, the image of $g(\mathbb{T})$ vanishes in $R_{O'}/(h(\mathbb{T}), l_i^{(j)})$ for every i .

For every $k \geq 1$, we have an injection

$$(R_{O'}/l_i^{(j-1)})/(l_{1,j}l_{2,j} \cdots l_{k,j}) \hookrightarrow \prod_{1 \leq i \leq k} (R_{O'}/l_i^{(j-1)})/(l_{i,j}) \cong \prod_{1 \leq i \leq k} R_{O'}/l_i^{(j)}$$

Since $R_{O'}/h(\mathbb{T})$ is finite flat over $O'[[W]]$, we get for each $k \geq 1$ an injection,

$$(R_{O'}/l_i^{(j-1)})/(h(\mathbb{T}), l_{1,j}l_{2,j} \cdots l_{k,j}) \hookrightarrow \prod_{1 \leq i \leq k} R_{O'}/(h(\mathbb{T}), l_i^{(j)}).$$

We observe that image of $g(\mathbb{T})$ vanishes in $(R_{O'}/l_i^{(j-1)})/(h(\mathbb{T}), l_{1,j}l_{2,j} \cdots l_{k,j})$. Thus the image of $g(\mathbb{T})$ is zero in $R_{O'}/(h(\mathbb{T}), l_i^{(j-1)})$, since

$$\varprojlim_k (R_{O'}/l_i^{(j-1)})/(h(\mathbb{T}), l_{1,j}l_{2,j} \cdots l_{k,j}) \cong R_{O'}/(h(\mathbb{T}), l_i^{(j-1)}).$$

For the μ invariants, for any l_i , $\pi_{O'}^\mu$ (resp. $\pi_{O'}^{\mu'}$) is equal to the highest power of π_O dividing the characteristic power series of $M_{O'}/l_i M_{O'}$ (resp. $N_{O'}/l_i N_{O'}$), where $M_{O'} := M \otimes_O O'$ is the extension of scalars from O to O' . Hence $\pi_{O'}^\mu = \pi_{O'}^{\mu'}$. Thus it follows that $\pi_{O'}^{\mu'}$ is zero in $(R_{O'}/l_i^{(j-1)})/\pi_{O'}^\mu$. \square

Proposition 3.7. *For any finitely generated $\mathcal{R}[[\Gamma]]$ module U , let U^0 denotes the maximal pseudonull $\mathcal{R}[[\Gamma]]$ submodule of U . Assume the hypothesis **(Tor)**. Then for every $\xi \in \mathfrak{X}(\mathcal{R})$, $\frac{\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})^0}{P_\xi \mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})^0}$ (resp. $\frac{\mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/F_{\text{cyc}})^0}{P_\xi \mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/F_{\text{cyc}})^0}$) are pseudonull $O_{f_\xi}[[\Gamma]]$ modules.*

Proof. We broadly follow the same strategy as in [Oc1, Lemma 7.2] but the arguments are different. We prove the result only for $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})$ as an entirely similar argument holds for $\mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/F_{\text{cyc}})$. Recall, for $C = \mathcal{A}$ or $C = A_{f_\xi}$, with notation as before, the Selmer group $S(C/F_{\text{cyc}})$ fits into the exact sequence

$$0 \rightarrow S(C/F_{\text{cyc}}) \rightarrow H^1(F_S/F_{\text{cyc}}, C) \rightarrow \bigoplus_{v_c | S, v_c \nmid p} H^1(I_{v_c}, C)^{G_{v_c}} \oplus \bigoplus_{v_c | \mathfrak{p}} H^1(I_{v_c}, C_{\mathfrak{p}}^-)^{G_{v_c}} \quad (42)$$

Note, for $v_c \nmid p$,

$$0 \rightarrow H^1(G_{v_c}/I_{v_c}, C^{I_{v_c}}) \rightarrow H^1(G_{v_c}, C) \rightarrow H^1(I_{v_c}, C)^{G_{v_c}} \rightarrow H^2(G_{v_c}/I_{v_c}, C^{I_{v_c}}) \quad (43)$$

with $G_{v_c}/I_{v_c} \cong \bigoplus_{l \neq p} \mathbb{Z}_l$ and C is a p -torsion group. Thus $H^i(G_{v_c}/I_{v_c}, C^{I_{v_c}}) = 0$ for $i = 1, 2$. Thus we see that, $H^1(G_{v_c}, C) \simeq H^1(I_{v_c}, C)^{G_{v_c}}$.

Combining these, we can have the following alternative definition of $S(C/F_{\text{cyc}})$.

$$0 \rightarrow S(C/F_{\text{cyc}}) \rightarrow H^1(F_S/F_{\text{cyc}}, C) \rightarrow \bigoplus_{v_c | S, v_c \nmid p} H^1(G_{v_c}, C) \oplus \bigoplus_{v_c | \mathfrak{p}} H^1(I_{v_c}, C_{\mathfrak{p}}^-)^{G_{v_c}} \quad (44)$$

Similarly, for strict Selmer group we obtain that,

$$0 \rightarrow S^{\text{str}}(C/F_{\text{cyc}}) \rightarrow H^1(F_S/F_{\text{cyc}}, C) \rightarrow \bigoplus_{v_c | S, v_c \nmid p} H^1(G_{v_c}, C) \oplus \bigoplus_{v_c | \mathfrak{p} | p} H^1(G_{v_c}, C_{\mathfrak{p}}^-) \quad (45)$$

We have from [F-O, §2.2.2] an exact sequence,

$$0 \rightarrow S^{\text{str}}(C/F_{\text{cyc}}) \rightarrow S(C/F_{\text{cyc}}) \rightarrow \bigoplus_{\mathfrak{p} | p} H^1(G_{F_{\mathfrak{p}}}/I_{\mathfrak{p}}, (C_{\mathfrak{p}}^-)^{I_{\mathfrak{p}}}).$$

Moreover from the proof of [F-O, Corollary 3.4], when $C = \mathcal{A}$, we see that $S^{\text{str}}(\mathcal{A}/F_{\text{cyc}}) = S(\mathcal{A}/F_{\text{cyc}})$.

Under the assumption **(Tor)**, we have that $X(T_{f_{\xi}}/F_{\text{cyc}})$ is torsion for any $\xi \in \mathfrak{X}(\mathcal{R})$. Also by corollary 3.5, $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})$ is torsion over $\mathcal{R}[[\Gamma]]$. It follows that the maps defining $S(C/F_{\text{cyc}})$ in (45) is surjective for $C = \mathcal{A}$ or $C = A_{f_{\xi}}$ with $\xi \in \mathfrak{X}(\mathcal{R})$ ([Oc1, Corollary 4.12]). Now let us consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & S^{\text{str}}(\mathcal{A}/F_{\text{cyc}}) & \longrightarrow & H^1(F_S/F_{\text{cyc}}, \mathcal{A}) & \longrightarrow & \bigoplus_{v_c | \mathfrak{p} | p} H^1(F_{\text{cyc}, v_c}, \mathcal{A}_{\mathfrak{p}}^-) \oplus \bigoplus_{v_c | S, v_c \nmid p} H^1(F_{\text{cyc}, v_c}, \mathcal{A}) \longrightarrow 0 \\ & & \uparrow \times P_{\xi} & & \uparrow \times P_{\xi} & & \uparrow \times P_{\xi} \\ 0 & \longrightarrow & S^{\text{str}}(\mathcal{A}/F_{\text{cyc}}) & \longrightarrow & H^1(F_S/F_{\text{cyc}}, \mathcal{A}) & \longrightarrow & \bigoplus_{v_c | \mathfrak{p} | p} H^1(F_{\text{cyc}, v_c}, \mathcal{A}_{\mathfrak{p}}^-) \oplus \bigoplus_{v_c | S, v_c \nmid p} H^1(F_{\text{cyc}, v_c}, \mathcal{A}) \longrightarrow 0. \end{array} \quad (46)$$

Recall $A_{f_{\xi}} \cong \mathcal{A}[P_{\xi}]$ and $(A_{f_{\xi}})_{\mathfrak{p}}^- \cong \mathcal{A}_{\mathfrak{p}}^-[P_{\xi}]$ as Galois modules. For any $\xi \in \mathfrak{X}(\mathcal{R})$, as $X(T_{f_{\xi}}^*/F_{\text{cyc}})$ is torsion by **(Tor)**, we get that $H^2(F_S/F_{\text{cyc}}, A_{f_{\xi}}) = 0$ (see [H-M, Proposition 2.3]). Then the cokernel of the middle vertical map in (46), being is a subgroup of $H^2(F_S/F_{\text{cyc}}, A_{f_{\xi}})$, vanishes. Thus by applying a Snake lemma to the diagram (46), we get that

$$\frac{S^{\text{str}}(\mathcal{A}/F_{\text{cyc}})}{P_{\xi} S^{\text{str}}(\mathcal{A}/F_{\text{cyc}})} \cong \text{coker}(H^1(F_S/F_{\text{cyc}}, \mathcal{A})[P_{\xi}] \xrightarrow{l_{\xi}} W[P_{\xi}]),$$

where

$$W := \bigoplus_{v_c | \mathfrak{p} | p} H^1(F_{\text{cyc}, v_c}, \mathcal{A}_{\mathfrak{p}}^-) \oplus \bigoplus_{v_c | S, v_c \nmid p} H^1(F_{\text{cyc}, v_c}, \mathcal{A}).$$

Similarly define

$$W_{\xi} := \bigoplus_{v_c | \mathfrak{p} | p} H^1(F_{\text{cyc}, v_c}, (A_{f_{\xi}})_{\mathfrak{p}}^-) \oplus \bigoplus_{v_c | S, v_c \nmid p} H^1(F_{\text{cyc}, v_c}, A_{f_{\xi}}).$$

Then we have the commutative diagram

$$0 \longrightarrow S^{\text{str}}(\mathcal{A}/F_{\text{cyc}})[P_{\xi}] \longrightarrow H^1(F_S/F_{\text{cyc}}, \mathcal{A})[P_{\xi}] \longrightarrow W[P_{\xi}] \longrightarrow \frac{S^{\text{str}}(\mathcal{A}/F_{\text{cyc}})}{P_{\xi} S^{\text{str}}(\mathcal{A}/F_{\text{cyc}})} \longrightarrow 0 \quad (47)$$

$$\begin{array}{ccccccc} & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & S^{\text{str}}(A_{f_{\xi}}/F_{\text{cyc}}) & \longrightarrow & H^1(F_S/F_{\text{cyc}}, A_{f_{\xi}}) & \longrightarrow & W_{\xi} \longrightarrow 0 \end{array}$$

We see that the natural map $W_\xi \longrightarrow W[P_\xi]$ above is surjective. From the diagram (47), we see that the natural map $H^1(F_S/F_{\text{cyc}}, \mathcal{A})[P_\xi] \xrightarrow{l_\xi} W[P_\xi]$ is surjective. Thus we obtain $S^{str}(\mathcal{A}/F_{\text{cyc}})/P_\xi S^{str}(\mathcal{A}/F_{\text{cyc}}) = 0$. Since $S^{str}(\mathcal{A}/F_{\text{cyc}}) = S(\mathcal{A}/F_{\text{cyc}})$, we see that, $S(\mathcal{A}/F_{\text{cyc}})/P_\xi S(\mathcal{A}/F_{\text{cyc}}) = 0$. In other words, $\mathcal{X}(\mathcal{T}_\mathcal{R}/F_{\text{cyc}})[P_\xi] = 0$. Thus $\mathcal{X}(\mathcal{T}_\mathcal{R}/F_{\text{cyc}})^0[P_\xi] = 0$. In particular, $\mathcal{X}(\mathcal{T}_\mathcal{R}/F_{\text{cyc}})^0[P_\xi]$ is a pseudonull $O_{f_\xi}[[\Gamma]]$ module. But for any finitely generated pseudonull $\mathcal{R}[[\Gamma]]$ module M , M/P_ξ is a pseudonull $O_{f_\xi}[[\Gamma]]$ module if and only if $M[P_\xi]$ is so ([Oc3, Lemma 3.1]). Thus, the last fact in turn implies that $\frac{\mathcal{X}(\mathcal{T}_\mathcal{R}/F_{\text{cyc}})^0}{P_\xi \mathcal{X}(\mathcal{T}_\mathcal{R}/F_{\text{cyc}})^0}$ is also a pseudonull $O_{f_\xi}[[\Gamma]]$ module. \square

Lemma 3.8. *Let J be any non-zero ideal in \mathcal{R} . Let $I \in \text{Spec}(\mathcal{R}) \setminus \text{Spec}(\mathcal{R}/J)$ (that is I is a prime ideal in \mathcal{R} , which does not contain J). There exists at most finitely many z_1, \dots, z_k in \mathcal{R} with the following properties:*

- (1) (I, z_i) are distinct prime ideals in \mathcal{R} for all $i = 1, \dots, k$.
- (2) For $i \neq j$, $\bar{z}_i \nmid \bar{z}_j$ in \mathcal{R}/I .
- (3) $(I, z_i) \supset J$ for all $i = 1, \dots, k$.

Proof. First we claim that,

$$\cap_{i=1}^r (I, z_i) = (I, z_1 z_2 \cdots z_r).$$

Obviously, $(I, z_1 z_2 \cdots z_r) \subseteq \cap_{i=1}^r (I, z_i)$. Now, let $x \in \cap_{i=1}^r (I, z_i)$. Then,

$$x = i_1 + a_1 z_1 = \cdots = i_r + a_r z_r,$$

with $i_j \in I$ and $a_j \in \mathcal{R}$. Then we see that,

$$\bar{x} = \bar{a}_1 \bar{z}_1 = \cdots = \bar{a}_r \bar{z}_r \in \mathcal{R}/I.$$

Since $\bar{z}_i \nmid \bar{z}_j$, we see that $\bar{x} = \bar{\alpha} \bar{z}_1 \cdots \bar{z}_r \in \mathcal{R}/I$. Thus, $x = i + \alpha z_1 \cdots z_r$, where $i \in I$ and α is some lift of $\bar{\alpha}$ in \mathcal{R} , which completes the proof of the claim.

Suppose that there are infinitely many z_i 's in \mathcal{R} which satisfies all three properties. Then we have a decreasing chain of ideals in \mathcal{R} ,

$$(I, z_1) \supset (I, z_1 z_2) = \cap_{i=1}^2 (I, z_i) \supset \cdots \supset (I, z_1 \cdots z_r) = \cap_{i=1}^r (I, z_i) \supset \cdots.$$

By assumption, (I, z_i) contains the ideal J for all i . Thus we obtain,

$$J \subseteq \cap_{i=1}^\infty (I, z_i) = \varprojlim_r (I, z_1 \cdots z_r) \cong I,$$

which is a contradiction to our assumption $I \in \text{Spec}(\mathcal{R}) \setminus \text{Spec}(\mathcal{R}/J)$. \square

Remark 3.9. Let $M = \mathcal{X}(\mathcal{T}_\mathcal{R}/F_{\text{cyc}})$ and $N = \mathcal{X}(\mathcal{T}_\mathcal{R}^*/F_{\text{cyc}})^\iota$. Assume **(Irr)**, **(Dist)**, **(Tor)** and $\mathcal{R} \cong O[[W]]$. Let S_1 be the subset of arithmetic points for which f_ξ 's are exceptional (as defined in Theorem 2.3). Also let S_2 be the subset of $\mathfrak{X}(\mathcal{R})$ for which at least one among $\ker(s_\xi)$, $\text{coker}(s_\xi)$, $\ker(s_\xi^*)$ and $\text{coker}(s_\xi^*)$ associated to the natural specialization map s_ξ in Theorem 3.1 and s_ξ^* in remark 3.2 is infinite. Define $S = S_1 \cup S_2$. Put $\mathfrak{X}(\mathcal{R})' := \mathfrak{X}(\mathcal{R}) \setminus S$, then $\mathfrak{X}(\mathcal{R})'$ is infinite.

Take the set $\{l_i\}_{i \in \mathbb{N}} = \mathfrak{X}(\mathcal{R})'$. Then from Theorem 3.1, corollary 3.5, Proposition 3.7 and Lemma 3.8, we deduce that for these choices of M , N and l_i 's, all the condition of Proposition 3.6 are satisfied (Lemma 3.8 ensures us that for each j in

the proof of Proposition 3.6, we have infinitely many ideals $l_i^{(j)}$ for which first $(j-1)$ generators are the same).

Theorem 3.10. *Let the notation be as before. Let F be a totally real number field, with $\Gamma = \text{Gal}(F_{\text{cyc}}/F) \cong \mathbb{Z}_p$. Assume*

- (1) **(Irr):** *The residual representation $\bar{\rho}_{\mathcal{R}}$ of G_F is absolutely irreducible.*
- (2) **(Dist):** *The restriction of the residual representation at the decomposition subgroup i.e. $\bar{\rho}_{\mathcal{R}}|_{G_{\mathfrak{p}}}$ is an extension of two distinct characters of $G_{\mathfrak{p}}$ with values in $\mathbb{F}_{\mathcal{R}}^*$ for each $\mathfrak{p}|p$.*
- (3) **(Tor):** *For any normalized cuspidal Hilbert eigenform f , $X(T_f/F_{\text{cyc}})$ is a finitely generated torsion $O_f[[\Gamma]]$ module.*
- (4) \mathcal{R} is a power series ring.

Then the functional equation holds for $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})$ i.e. as an ideal in $\mathcal{R}[[\Gamma]]$,

$$\text{Ch}_{\mathcal{R}[[\Gamma]]}(\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})) = \text{Ch}_{\mathcal{R}[[\Gamma]]}(\mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/F_{\text{cyc}})^{\iota}).$$

Proof. By corollary 3.5, $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})$ and $\mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/F_{\text{cyc}})$ are torsion $\mathcal{R}[[\Gamma]]$ modules. Using remark 3.9, choose the infinite subset $\mathfrak{X}(\mathcal{R})'$ of arithmetic points. By corollary 3.5, for every $\xi \in \mathfrak{X}(\mathcal{R})'$, $X(T_{f_{\xi}}/F_{\text{cyc}})$ and $X(T_{f_{\xi}}^*/F_{\text{cyc}})^{\iota}$ are torsion over $O_{f_{\xi}}[[\Gamma]]$. Then applying Proposition 3.6 for $M = \mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})$, $N = \mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/F_{\text{cyc}})^{\iota}$ and $\{l_i\}_{i \in \mathbb{N}} = \mathfrak{X}(\mathcal{R})'$, to prove the theorem it suffices to show that for every $\xi \in \mathfrak{X}(\mathcal{R})'$,

$$\text{Ch}_{O_{f_{\xi}}[[\Gamma]]}\left(\frac{\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})}{P_{\xi}\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})}\right) = \text{Ch}_{O_{f_{\xi}}[[\Gamma]]}\left(\frac{\mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/F_{\text{cyc}})^{\iota}}{P_{\xi}\mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/F_{\text{cyc}})^{\iota}}\right)$$

considered as ideals in $O_{f_{\xi}}[[\Gamma]]$. By Theorem 3.1, remark 3.2 and remark 3.3 this in turn equivalent to showing

$$\text{Ch}_{O_{f_{\xi}}[[\Gamma]]}(X(T_{f_{\xi}}/F_{\text{cyc}})) = \text{Ch}_{O_{f_{\xi}}[[\Gamma]]}(X(T_{f_{\xi}}^*/F_{\text{cyc}})^{\iota})$$

for each $\xi \in \mathfrak{X}(\mathcal{R})'$. Hence we are done by Theorem 2.10. \square

4. RESULTS OVER \mathbb{Z}_p^2 EXTENSION

Let K_{∞}/K be the unique $\mathbb{Z}_p^{\oplus 2}$ extension of an imaginary quadratic field K . In this section, we will assume throughout that p splits in K and D_K the discriminant of the imaginary quadratic field K , is coprime to tame conductor $N_{\mathcal{R}}$ of the branch \mathcal{R} of the Hida family i.e. $(p, D_K) = (D_K, N_{\mathcal{R}}) = (N_{\mathcal{R}}, p) = 1$. Recall the notation, $\Gamma_K = \text{Gal}(K_{\infty}/K) \cong \mathbb{Z}_p^2$, $\Gamma = G(K_{\text{cyc}}/K) \cong \mathbb{Z}_p$ and $H = \text{Gal}(K_{\infty}/K_{\text{cyc}}) \cong \mathbb{Z}_p$ so that $G/H \cong \Gamma$.

Recall from remark 1.21, $S = S_K$ is a finite set of primes of K dividing Np . Let v be a prime of K in S . Denote by v_c a prime of K_{cyc} lying above v and let v_{∞} be a prime of K_{∞} lying above v_c . Let \bar{v} be a prime of $\bar{\mathbb{Q}}$ lying above a prime v_{∞} . Let G_{v_c} and $G_{v_{\infty}}$ denotes the decomposition subgroup of $\bar{\mathbb{Q}}/K_{\text{cyc}}$ and $\bar{\mathbb{Q}}/K_{\infty}$ for the prime \bar{v}/v_c and \bar{v}/v_{∞} respectively. Let I_{v_c} , $I_{v_{\infty}}$ denote the inertia group of $\bar{\mathbb{Q}}/K_{\text{cyc}}$ and $\bar{\mathbb{Q}}/K_{\infty}$ for the prime \bar{v}/v_c and \bar{v}/v_{∞} respectively. Let Γ_{K_v} (resp. H_{v_c}) denote the decomposition subgroup of Γ_K (resp. H) with respect to primes v_{∞}/v (resp. v_{∞}/v_c). We write I_{v_{∞}/v_c} for the inertia subgroup of $K_{\infty}/K_{\text{cyc}}$ with respect to prime v_{∞}/v_c .

Proposition 4.1. *The kernel of the map*

$$X(T_f/K_\infty)_H \xrightarrow{\alpha_H^\vee} X(T_f/K_{\text{cyc}})$$

is a finitely generated \mathbb{Z}_p module and the cokernel of α_H^\vee is finite.

Proof. Set

$$J_{v_c} := \begin{cases} H^1(I_{v_c}, A_f)^{G_{v_c}} & \text{if } v_c \mid S, v_c \nmid p \\ H^1(I_{v_c}, A_f^-)^{G_{v_c}} & \text{if } v_c \mid p, \end{cases}$$

$$J_{v_c}^\infty := \begin{cases} \prod_{v_\infty \mid v_c} H^1(I_{v_\infty}, A_f)^{G_{v_\infty}} & \text{if } v_c \mid S, v_c \nmid p \\ \prod_{v_\infty \mid v_c} H^1(I_{v_\infty}, A_f^-)^{G_{v_\infty}} & \text{if } v_c \mid p, \end{cases}$$

Using similar argument as in (43), we get that

$$J_{v_c} := \begin{cases} H^1(K_{\text{cyc}, v_c}, A_f) & \text{if } v_c \mid S, v_c \nmid p \\ H^1(I_{v_c}, A_f^-)^{G_{v_c}} & \text{if } v_c \mid p, \end{cases}$$

Also,

$$J_{v_c}^\infty := \begin{cases} \prod_{v_\infty \mid v_c} H^1(K_{\infty, v_\infty}, A_f) \cong (\text{Ind}_{H_{v_c}}^H H^1(K_{\infty, v_\infty}, A_f)^\vee)^\vee & \text{if } v_c \mid S, v_c \nmid p \\ \prod_{v_\infty \mid v_c} H^1(I_{v_\infty}, A_f^-)^{G_{v_\infty}} \cong (\text{Ind}_{H_{v_c}}^H H^1(I_{v_\infty}, A_f)^{G_{v_\infty}})^\vee & \text{if } v_c \mid p, \end{cases}$$

Then we have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & S(A_f/K_\infty)^H & \longrightarrow & H^1(K_S/K_\infty, A_f)^H & \longrightarrow & (\prod_{v_c \mid S} J_{v_c}^\infty)^H \\ & & \uparrow \alpha_H & & \uparrow \phi_H & & \uparrow \delta_H = \prod \delta^{v_c} \\ 0 & \longrightarrow & S(A_f/K_{\text{cyc}}) & \longrightarrow & H^1(K_S/K_{\text{cyc}}, A_f) & \longrightarrow & \prod_{v_c \mid S} J_{v_c} \end{array} \quad (48)$$

By inflation-restriction sequence of Galois cohomology, the kernel of ϕ_H is isomorphic to $H^1(H, A_f^{G_{K_\infty}})$. Clearly $U_\infty := A_f^{G_{K_\infty}^\vee}$ is a finitely generated \mathbb{Z}_p module. Thus we deduce that $\ker(\phi_H)^\vee$ is a finitely generated \mathbb{Z}_p module. Moreover, as $H \cong \mathbb{Z}_p$, we see that $H^1(H, A_f^{G_{K_\infty}})$ is finite if and only if $H^0(H, A_f^{G_{K_\infty}}) \cong A_f^{G_{K_{\text{cyc}}}}$ is finite. The last fact follows from (cf. [C-S, Theorem A 2.8], [Su, Lemma 2.1]). Hence, by Snake lemma on diagram (48), we deduce that $\text{cocker}(\alpha_H^\vee)$ is finite.

Also as H has p -cohomological dimension $= 1$, ϕ_H is surjective. Thus $\ker(\alpha_H)^\vee$ is a subquotient of the Pontryagin dual of kernel of δ_H . Given a prime v_c we pick any one prime v_∞ in K_∞ . Then by Shapiro's Lemma for each v_c ,

$$H^1(H, J_{v_c}^\infty) \cong \begin{cases} H^1(K_{\infty, v_\infty}, A_f) & \text{if } v_c \mid S, v_c \nmid p \\ H^1(I_{v_\infty}, A_f^-)^{G_{v_\infty}} & \text{if } v_c \mid p, \end{cases}$$

Then

$$\ker(\delta^{v_c}) \cong \begin{cases} H^1(H_{v_c}, A_f^{G_{v_\infty}}) & \text{if } v_c \mid S, v_c \nmid p \\ \text{a subgroup of } H^1(I_{v_\infty/v_c}, A_f^{-I_{v_\infty}}) & \text{if } v_c \mid p, \end{cases}$$

where the inertia subgroup $I_{v_\infty/v_c} \cong \mathbb{Z}_p$. As before, we conclude that the Pontryagin dual of $\ker(\delta^{v_c})$ is a finitely generated \mathbb{Z}_p module. From these, summing over finitely many primes and using a Snake lemma on diagram (48), we deduce that $\ker(\alpha_H^\vee)$ is finitely generated over \mathbb{Z}_p . \square

Remark 4.2. By Kato's result (see [Ka]), we know that for any p -stabilized newform f , $X(T_f/K_{\text{cyc}})$ (and $X(T_f^*/K_{\text{cyc}})$) are finitely generated torsion $\mathbb{Z}_p[[\Gamma]]$ modules.

Corollary 4.3. For any p -stabilized newform f , $X(T_f/K_\infty)$ and $X(T_f^*/K_\infty)$ are finitely generated **torsion** $O_f[[\Gamma_K]]$ modules.

Proof. By remark 4.2, we have $X(T_f/K_{\text{cyc}})$ is a finitely generated torsion $O_f[[\Gamma]]$ module. By Proposition 4.1, we also have an exact sequence

$$0 \longrightarrow F_1 \longrightarrow X(T_f/K_\infty)_H \longrightarrow X(T_f/K_{\text{cyc}}) \longrightarrow F_2 \longrightarrow 0$$

with F_1 is a finitely generated \mathbb{Z}_p modules and F_2 is finite. Thus we get that $X(T_f/K_\infty)_H$ is a finite generated torsion $O_f[[\Gamma]]$ module. But for an ideal $I \neq O_f[[\Gamma_K]]$ in $O_f[[\Gamma_K]]$

$$\text{rank}_{\frac{O_f[[\Gamma_K]]}{I}} \frac{X(T_f/K_\infty)}{IX(T_f/K_\infty)} \geq \text{rank}_{O_f[[\Gamma_K]]} X(T_f/K_\infty).$$

Identifying $O_f[[\Gamma_K]]$ with $O_f[[T_1, T_2]]$ and $X(T_f/K_\infty)_H$ with $\frac{X(T_f/K_\infty)}{T_2 X(T_f/F_\infty)}$, we deduce that $\text{rank}_{O_f[[\Gamma_K]]} X(T_f/K_\infty) = 0$. The argument for $X(T_f^*/K_\infty)$ is similar. \square

Definition 4.4. We call $f \in S_2(\Gamma_0(Np), \psi)$ *exceptional* if f is a newform of conductor Np with $(N, p) = (\text{conductor of } \psi, p) = 1$.

Theorem 4.5. Let the notation be as before. Let $f \in S_k(\Gamma_0(Np^r), \psi)$ be a p -stabilized newform which is not exceptional. Also assume that $(N, D_K) = (p, D_K) = 1$. Then the functional equation holds for $X(T_f/K_\infty)$ i.e. we have a equality of ideals in $O_f[[\Gamma_K]]$,

$$Ch_{O_f[[\Gamma_K]]}(X(T_f/K_\infty)) = Ch_{O_f[[\Gamma_K]]}(X(T_f^*/K_\infty)^\iota).$$

Proof. Note as the p -ordinary, p -stabilized newform f is not exceptional and we have, $(N, D_K) = (p, D_K) = 1$; the Galois representation (ρ_f, V_f) satisfies both $(\text{Hyp}(K_\infty, V_f))$ and $(\text{Hyp}(K_\infty, V_f^\vee))$ assumptions of [Pe, Theorem 4.2.1]. Also, by corollary 4.3, we get that both $X(T_f/K_\infty)$ and $X(T_f^*/K_\infty)^\iota$ are torsion over $O_f[[\Gamma_K]]$. Thus the condition $(\text{Tors}(K_\infty, V_f))$ in [Pe, Theorem 4.2.1] is also satisfied. Hence the theorem follows from [Pe, Theorem 4.2.1]. \square

Let us recall from remark 1.12, $\mathcal{T}_{\mathcal{R}}$ is a lattice associated to a fixed branch \mathcal{R} of the ordinary Hida family of elliptic modular form of tame level $N = N_{\mathcal{R}}$.

Proposition 4.6. *Assume (Irr), (Dist) and \mathcal{R} is a power series ring. Then the kernel and the cokernel of the natural specialization map*

$$\mathcal{X}(\mathcal{T}_{\mathcal{R}}/K_{\infty})/P_{\xi}\mathcal{X}(\mathcal{T}_{\mathcal{R}}/K_{\infty}) \xrightarrow{\beta_{\xi}^{\vee}} X(T_{f_{\xi}}/K_{\infty}) \quad (49)$$

are pseudonull $O_{f_{\xi}}[[\Gamma_K]] \cong O_{f_{\xi}}[[T_1, T_2]]$ module for all but finitely many $\xi \in \mathfrak{X}(\mathcal{R})$. In particular, the equality

$$Ch_{O_{f_{\xi}}[[\Gamma_K]]}(\mathcal{X}(\mathcal{T}_{\mathcal{R}}/K_{\infty})/P_{\xi}\mathcal{X}(\mathcal{T}_{\mathcal{R}}/K_{\infty})) = Ch_{O_{f_{\xi}}[[\Gamma_K]]}(X(T_{f_{\xi}}/K_{\infty}))$$

as ideals in $O_{f_{\xi}}[[\Gamma_K]]$ holds for all but finitely many arithmetic points.

Proof. The proof is different from the proof of Theorem 3.1 in two points. Here, we have to overcome the difficulty that there are possibly infinitely many primes in K_{∞} lying above a given prime in K . On the other hand, here we have the advantage that $\mathcal{R} \cong O[[W]]$, so that P_{ξ} is principal.

Let us keep the notation as set up in the beginning of section 4. For a finitely generated $O_{f_{\xi}}[[\Gamma_{K_v}]]$ module M , recall $\text{Ind}_{\Gamma_{K_v}}^{\Gamma_K} M := O_{f_{\xi}}[[\Gamma_K]] \otimes_{O_{f_{\xi}}[[\Gamma_{K_v}]]} M$. Recall, by Shapiro's lemma $H_i(\Gamma_K, \text{Ind}_{\Gamma_{K_v}}^{\Gamma_K} M) \cong H_i(\Gamma_{K_v}, M)$ for any $i \geq 0$.

Now, we have the commutative diagram with the natural maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & S(\mathcal{A}/K_{\infty})[P_{\xi}] & \longrightarrow & H^1(K_S/K_{\infty}, \mathcal{A})[P_{\xi}] & \longrightarrow & \prod_{v \in S} J_v(\mathcal{A})[P_{\xi}] \\ & & \uparrow \beta_{\xi} & & \uparrow \eta_{\xi} & & \uparrow \delta_{\xi} = \prod \delta_v \\ 0 & \longrightarrow & S(A_{f_{\xi}}/K_{\infty}) & \longrightarrow & H^1(K_S/K_{\infty}, A_{f_{\xi}}) & \longrightarrow & \prod_{v \in S} J_v(A_{f_{\xi}}). \end{array} \quad (50)$$

Here

$$J_v(B) := \begin{cases} (\text{Ind}_{\Gamma_{K_v}}^{\Gamma_K} H^1(I_{v_{\infty}}, B)_{G_{v_{\infty}}}^{\vee})^{\vee} & \text{if } v \in S, v \nmid p \\ (\text{Ind}_{\Gamma_{K_v}}^{\Gamma_K} H^1(I_{v_{\infty}}, B^-)_{G_{v_{\infty}}}^{\vee})^{\vee} & \text{if } v \nmid p, \end{cases}$$

for $B = \mathcal{A}$ or $B = A_{f_{\xi}}$. Recall, once again $A_{f_{\xi}} \cong \mathcal{A}[P_{\xi}]$ and $A_{f_{\xi}}^- \cong \mathcal{A}^-[P_{\xi}]$. Also by our assumption that \mathcal{R} is a power series ring, we have every P_{ξ} is principal ideal in \mathcal{R} . Then η_{ξ} is surjective with $\ker(\eta_{\xi}) \cong \mathcal{A}^{G_{K_{\infty}}}/P_{\xi}\mathcal{A}^{G_{K_{\infty}}}$. To simplify notation, we put $\mathcal{T} := \mathcal{T}_{\mathcal{R}}$ in the proof of this theorem. Then $\ker(\eta_{\xi})^{\vee} \cong (\mathcal{T}_{G_{K_{\infty}}}^{\dagger})_{\text{tor}}[P_{\xi}]$. Note, only finitely many P_{ξ} divide the \mathcal{R} characteristic ideal of finitely generated torsion \mathcal{R} module $(\mathcal{T}_{G_{K_{\infty}}}^{\dagger})_{\text{tor}}$. Hence we deduce that the $\ker(\beta_{\xi})^{\vee}$ is finite (hence $O_{f_{\xi}}[[\Gamma_K]]$ pseudonull) for all but finitely many $\xi \in \mathfrak{X}(\mathcal{R})$.

As before, it suffices to show that $\ker(\delta_{\xi})^{\vee}$ is $O_{f_{\xi}}[[\Gamma_K]]$ pseudonull leaving out finitely many exceptional ξ . Now it is easy to see that

$$(\ker(\delta_v))^{\vee} = \begin{cases} \text{Ind}_{\Gamma_{K_v}}^{\Gamma_K} ((\frac{\mathcal{A}^{I_{v_{\infty}}}}{P_{\xi}\mathcal{A}^{I_{v_{\infty}}}})^{G_{v_{\infty}}})^{\vee} \cong \text{Ind}_{\Gamma_{K_v}}^{\Gamma_K} (\mathcal{T}_{I_{v_{\infty}}}^{\dagger}[P_{\xi}])_{G_{v_{\infty}}} & \text{if } v \in S, v \nmid p \\ \text{Ind}_{\Gamma_{K_v}}^{\Gamma_K} ((\frac{\mathcal{A}^{-I_{v_{\infty}}}}{P_{\xi}\mathcal{A}^{-I_{v_{\infty}}}})^{G_{v_{\infty}}})^{\vee} \cong \text{Ind}_{\Gamma_{K_v}}^{\Gamma_K} (\mathcal{T}_{I_{v_{\infty}}}^{-\dagger}[P_{\xi}])_{G_{v_{\infty}}} & \text{if } v \nmid p, \end{cases}$$

But for a prime v in S not lying above p , we have $\mathcal{T}_{I_{v_{\infty}}}^{\dagger}[P_{\xi}] \cong (\mathcal{T}_{I_{v_{\infty}}}^{\dagger})_{\text{tor}}[P_{\xi}]$ which is, as before, finite for all but finitely many ξ . Notice that K_{∞} contains K_{cyc} and hence the dimension of Γ_{K_v} as a p -adic Lie group is at least one. Hence Krull dimension

of the commutative ring $O_{f_\xi}[[\Gamma_{K_v}]]$ is at least 2. Thus $(\mathcal{T}_{I_{v\infty}}^\dagger[P_\xi])_{G_{K_\infty, v\infty}}$ is finite and hence pseudonull as an $O_{f_\xi}[[\Gamma_{K_v}]]$ module. Also, for a finitely generated $O_{f_\xi}[[\Gamma_{K_v}]]$ module M and for $i = 0, 1$ we have [Ve, Lemma 2.7(i)]

$$\text{Ext}_{O_{f_\xi}[[\Gamma_K]]}^i(O_{f_\xi}[[\Gamma_K]] \otimes_{O_{f_\xi}[[\Gamma_{K_v}]]} M, O_{f_\xi}[[\Gamma_K]]) \cong O_{f_\xi}[[\Gamma_K]] \otimes_{O_{f_\xi}[[\Gamma_{K_v}]]} \text{Ext}_{O_{f_\xi}[[\Gamma_{K_v}]]}^i(M, O_{f_\xi}[[\Gamma_{K_v}]]).$$

But a finitely generated Λ (for $\Lambda = O_{f_\xi}[[\Gamma_K]]$ or $O_{f_\xi}[[\Gamma_{K_v}]]$) module M is pseudonull if and only if $\text{Ext}_\Lambda^i(M, \Lambda) = 0$ for $i = 0, 1$ ([Ve]). Thus we see that $\text{Ind}_{\Gamma_{K_v}}^{\Gamma_K}(\mathcal{T}_{I_{v\infty}}^\dagger[P_\xi])_{G_{v\infty}}$ is pseudonull as a $O_{f_\xi}[[\Gamma_K]]$ module for all but finitely many ξ and for any $v \in S, v \nmid p$. The same argument holds for a prime v in S dividing p if we replace \mathcal{T} by \mathcal{T}^- . Combining these for finitely many v in S , we deduce that $(\ker(\delta_\xi))^\vee$ is a pseudonull $O_\xi[[\Gamma_K]]$ module for all but finitely many $\xi \in \mathfrak{X}(\mathcal{R})$. This completes the proof. \square

Remark 4.7. We have analogous of remarks 3.2 and 3.3 for K_∞ for the map $\mathcal{X}(\mathcal{T}_\mathcal{R}^*/K_\infty)/P_\xi \mathcal{X}(\mathcal{T}_\mathcal{R}^*/K_\infty) \xrightarrow{\beta_\xi^*} X(T_{f_\xi}/K_\infty)$. Proceeding in Proposition 4.6 and using Corollary 4.3 we can get,

$$\text{Ch}_{O_{f_\xi}[[\Gamma_K]]}(\mathcal{X}(\mathcal{T}_\mathcal{R}^*/K_\infty)/P_\xi \mathcal{X}(\mathcal{T}_\mathcal{R}^*/K_\infty)) = \text{Ch}_{O_{f_\xi}[[\Gamma_K]]}(X(T_{f_\xi}^*/K_\infty))$$

as ideals in $O_{f_\xi}[[\Gamma_K]]$ for all but finitely many $\xi \in \mathfrak{X}(\mathcal{R})$. Further, applying the involution ι ,

$$\text{Ch}_{O_{f_\xi}[[\Gamma_K]]}(\mathcal{X}(\mathcal{T}_\mathcal{R}^*/K_\infty)^\iota/P_\xi \mathcal{X}(\mathcal{T}_\mathcal{R}^*/K_\infty)^\iota) = \text{Ch}_{O_{f_\xi}[[\Gamma_K]]}(X(T_{f_\xi}^*/K_\infty)^\iota)$$

holds. Moreover, proceeding as in Corollary 3.5, we deduce that both $\mathcal{X}(\mathcal{T}_\mathcal{R}/K_\infty)$ and $\mathcal{X}(\mathcal{T}_\mathcal{R}^*/K_\infty)$ are finitely generated **torsion** modules over $\mathcal{R}[[\Gamma_K]]$.

Proposition 4.8. For any finitely generated $\mathcal{R}[[\Gamma_K]]$ module U , let U^0 denotes the maximal pseudonull $\mathcal{R}[[\Gamma_K]]$ submodule of U . Then $\frac{\mathcal{X}(\mathcal{T}_\mathcal{R}/K_\infty)^0}{P_\xi \mathcal{X}(\mathcal{T}_\mathcal{R}/K_\infty)^0}$ and $\frac{\mathcal{X}(\mathcal{T}_\mathcal{R}^*/K_\infty)^0}{P_\xi \mathcal{X}(\mathcal{T}_\mathcal{R}^*/K_\infty)^0}$ are pseudonull $O_{f_\xi}[[\Gamma_K]]$ modules for any $\xi \in \mathfrak{X}(\mathcal{R})$.

Proof. We proceed as in the proof of Proposition 3.7. Here the proof is different from Proposition 3.7 as the strict Selmer group and the usual Selmer group for \mathcal{A} over K_∞ may not coincide. As before, we will prove for $\mathcal{X}(\mathcal{T}_\mathcal{R}/K_\infty)$ only.

As in Proposition 3.7, it suffices to show $\mathcal{X}(\mathcal{T}_\mathcal{R}/K_\infty)[P_\xi] = 0$ for every ξ . for $B = \mathcal{A}$ or A_{f_ξ} , recall from section 1.2, the strict Selmer group $S'(B/K_\infty) \hookrightarrow S(B/K_\infty)$. Then by corollary 4.3 and remark 4.7, we deduce that all four groups $X(T_{f_\xi}/K_\infty)$, $X'(T_{f_\xi}/K_\infty)$, $\mathcal{X}(\mathcal{T}_\mathcal{R}/K_\infty)$ and $\mathcal{X}'(\mathcal{T}_\mathcal{R}/K_\infty)$ are finitely generated torsion modules over their respective Iwasawa algebras $O_{f_\xi}[[\Gamma_K]]$ and $\mathcal{R}[[\Gamma_K]]$. It follows that the maps defining the Greenberg Selmer groups and strict Selmer groups over K_∞ are surjective ([Oc1, Theorem 4.10], [H-V, Theorem 7.12]). In other words, $\phi_{K_\infty}^B$ is surjective for $B = \mathcal{A}$ or $B = A_{f_\xi}$ and we have the exact sequence

$$0 \longrightarrow S(B/K_\infty) \longrightarrow H^1(K_S/K_\infty, B) \longrightarrow \prod_{v \in S, v \nmid p} J_v(B/K_\infty) \prod_{v \mid p} J_v^p(B^-/K_\infty) \longrightarrow 0$$

Here $J_v(B/K_\infty) = (\text{Ind}_{G_v}^G H^1(K_{\infty, v\infty}, B)^\vee)^\vee$ and $J_v^p(B^-/K_\infty) = (\text{Ind}_{G_v}^G H^1(I_{v\infty}, B)^{G_{v\infty}})^\vee$. Hence we get another exact sequence

$$0 \longrightarrow S'(B/K_\infty) \longrightarrow S(B/K_\infty) \longrightarrow U_p(B) \longrightarrow 0$$

with $U_p(B) := (\text{Ind}_{\Gamma_{K_v}}^{\Gamma_K} H^1(G_{v_\infty}/I_{v_\infty}, (B^-)^{I_{v_\infty}})^\vee)^\vee$. Therefore, it suffices to show that $\mathcal{X}'(\mathcal{T}_\mathcal{R}/F_\infty)[P_\xi] = 0$ and $U_p(\mathcal{A})^\vee[P_\xi] = 0$. Now,

$$H_1(G_{v_\infty}/I_{v_\infty}, \mathcal{A}^{-\vee})[P_\xi] = H_1(G_{v_\infty}/I_{v_\infty}, \mathcal{T}_\mathcal{R}^{\dagger-})[P_\xi], \quad (51)$$

where $\mathcal{T}_\mathcal{R}^{\dagger-}$ is a free \mathcal{R} module of rank 1. As before, write $\langle \text{Fr}_{v_\infty} \rangle := G_{v_\infty}/I_{v_\infty}$. Then the module in (51) is isomorphic to $(\mathcal{T}_\mathcal{R}^{\dagger-}[\text{Fr}_{v_\infty}])[P_\xi]$. But $(\mathcal{T}_\mathcal{R}^{\dagger-}[\text{Fr}_{v_\infty}])[P_\xi] \subset (\mathcal{T}_\mathcal{R}^{\dagger-})[P_\xi] = 0$ as $\mathcal{T}_\mathcal{R}^{\dagger-}$ is a free \mathcal{R} module. Using this, it is plain from the definition of $U_p(\mathcal{A})$ that $U_p(\mathcal{A})^\vee[P_\xi] = 0$.

We will show $\mathcal{X}'(\mathcal{T}_\mathcal{R}/K_\infty)[P_\xi] = 0$ to complete the proof of the proposition. Set

$$W'_\xi := \prod_{v \in S, v \nmid p} (\text{Ind}_{\Gamma_{K_v}}^{\Gamma_K} H^1(K_{\infty, v_\infty}, A_{f_\xi})^\vee)^\vee \prod_{v|p} (\text{Ind}_{\Gamma_{K_v}}^{\Gamma_K} H^1(K_{\infty, v_\infty}, A_{f_\xi}^-)^\vee)^\vee \quad \text{and}$$

$$\mathcal{W}' := \prod_{v \in S, v \nmid p} (\text{Ind}_{\Gamma_{K_v}}^{\Gamma_K} H^1(K_{\infty, v_\infty}, \mathcal{A})^\vee)^\vee \prod_{v|p} (\text{Ind}_{\Gamma_{K_v}}^{\Gamma_K} H^1(K_{\infty, v_\infty}, \mathcal{A}^-)^\vee)^\vee \quad \text{and}$$

These are the ‘local factors’ used in the definition of $X'(T_{f_\xi}/K_\infty)$ and $\mathcal{X}'(\mathcal{T}_\mathcal{R}/K_\infty)$ respectively. Once again, as in proposition 3.7, this follows if we can show the map $W'_\xi \rightarrow \mathcal{W}'[P_\xi]$ is surjective. Recall as $A[P_\xi] \cong A_{f_\xi}$. But for every prime v_∞ of K_∞ lying above a prime S in K not above p , the map $H^1(K_{\infty, v_\infty}, \mathcal{A}[P_\xi]) \rightarrow H^1(K_{\infty, v_\infty}, \mathcal{A})[P_\xi]$ is surjective by Kummer theory. Similarly, for every $v_\infty \mid p$, $H^1(K_{\infty, v_\infty}, A_{f_\xi}^-) \rightarrow H^1(K_{\infty, v_\infty}, \mathcal{A}^-)[P_\xi]$ is surjective. From these, it follows that $W'_\xi \rightarrow \mathcal{W}'[P_\xi]$ is indeed surjective. This completes the proof of the proposition. \square

We now state our main theorem in the \mathbb{Z}_p^2 extension case.

Theorem 4.9. *Let the notation be as before. Let K be an imaginary quadratic field and $K \subset K_{\text{cyc}} \subset K_\infty$ be the unique $\mathbb{Z}_p^{\oplus 2}$ extension of K . Assume*

- (1) $(N_\mathcal{R}, D_K) = (p, D_K) = 1$ i.e. p splits in K and the tame conductor $N_\mathcal{R}$ associated to the branch of the Hida family is coprime to the discriminant of K .
- (2) **(Irr)** : The residual representation $\bar{\rho}_\mathcal{R}$ is absolutely irreducible as $G_\mathbb{Q}$ -module.
- (3) **(Dist)** : The characters appearing on the diagonal of the local residual representation $\bar{\rho}_\mathcal{R}|_{G_p}$ are distinct (mod \mathfrak{m}).
- (4) \mathcal{R} is a power series ring.

Then the functional equation holds for $\mathcal{X}(\mathcal{T}_\mathcal{R}/K_\infty)$ i.e. as an ideal in $\mathcal{R}[[\Gamma_K]]$,

$$\text{Ch}_{\mathcal{R}[[\Gamma_K]]}(\mathcal{X}(\mathcal{T}_\mathcal{R}/K_\infty)) = \text{Ch}_{\mathcal{R}[[\Gamma_K]]}(\mathcal{X}(\mathcal{T}_\mathcal{R}^*/K_\infty)^\iota)$$

Proof. By remark 4.7, we see that $\mathcal{X}(\mathcal{T}_\mathcal{R}/K_\infty)$ and $\mathcal{X}(\mathcal{T}_\mathcal{R}^*/K_\infty)$ are finitely generated torsion $\mathcal{R}[[\Gamma_K]]$ module. Let S_1 be the finite subset of $\mathfrak{X}(\mathcal{R})$ consisting of those ξ for which f_ξ is exceptional. Let S_2 be the finite subset of $\mathfrak{X}(\mathcal{R})$ for which the map β_ξ in Proposition 4.6 is not a pseudo-isomorphism. Similarly S_3 be the finite subset of $\mathfrak{X}(\mathcal{R})$ for which the map β_ξ^* in the remark 4.7 is not a pseudo-isomorphism. Let

$\mathfrak{S} = S_1 \cup S_2 \cup S_3$ be the finite subset of $\mathfrak{X}(\mathcal{R})$. Then by Proposition 3.6, to prove the theorem, it suffices to show that for all $\xi \in \mathfrak{X}(\mathcal{R}) \setminus \mathfrak{S}$,

$$Ch_{O_{f_\xi}[[\Gamma_K]]}(\mathcal{X}(\mathcal{T}_{\mathcal{R}}/K_\infty)/P_\xi \mathcal{X}(\mathcal{T}_{\mathcal{R}}/K_\infty)) = Ch_{O_{f_\xi}[[\Gamma_K]]}(\mathcal{X}(\mathcal{T}_{\mathcal{R}}^*/K_\infty)^\iota/P_\xi \mathcal{X}(\mathcal{T}_{\mathcal{R}}/K_\infty)^\iota). \quad (52)$$

Using Proposition 4.6 and remark 4.7, (52) is in turn equivalent to showing for all $\xi \in \mathfrak{X}(\mathcal{R}) \setminus \mathfrak{S}$,

$$Ch_{O_{f_\xi}[[\Gamma_K]]}(X(\mathcal{T}_{f_\xi}/K_\infty)) = Ch_{O_{f_\xi}[[\Gamma_K]]}(X(\mathcal{T}_{f_\xi}^*/K_\infty)^\iota), \quad (53)$$

which follows from Theorem 4.5. This completes the proof of the proposition. \square

Remark 4.10. Our proof of Theorem 3.10 (respectively Theorem 4.9) covers the case when $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/F_{\text{cyc}})$ (respectively $\mathcal{X}(\mathcal{T}_{\mathcal{R}}/K_\infty)$) have non zero $\mathcal{R}[[\Gamma]]$ (respectively $\mathcal{R}[[\Gamma_K]]$) pseudonull submodules. We also allow the Selmer groups $X(T_{f_\xi}/F_{\text{cyc}})$, $X(T_{f_\xi}/K_{\text{cyc}})$, $\xi \in \mathfrak{X}(\mathcal{R})$ to have positive μ -invariant. Our proof of Theorem 4.9 works for both CM or non-CM Hida family.

Remark 4.11. Following Mazur's conjecture, the crucial assumption (Tor) for Hilbert modular forms in Theorem 3.10, is expected to be always true but has not been proven except in a few special cases. All the other conditions in Theorem 3.10 (respectively all the conditions in Theorem 4.9) are satisfied in many cases.

Remark 4.12. Keeping Leopoldt conjecture for a totally real field F in mind, we do not pursue the \mathbb{Z}_p^2 extension case in case of Hilbert modular forms.

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SOMNATH JHA, SCHOOL OF PHYSICAL SCIENCES, JAWAHARLAL NEHRU UNIVERSITY, NEW DELHI - 110067, INDIA

E-mail address: jhasomnath@gmail.com

DIPRAMIT MAJUMDAR, INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH PUNE, PASHAN, PUNE 411008, MAHARASHTRA, INDIA

E-mail address: dipramit@gmail.com